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ABSTRACT

This guide represents the final experimental version of a pilot project which was conducted in the United States between 1973 and 1976. The ideas and the manner of presentation are based on the works of Georges and Frederique Papy. They are recognized for having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. This program follows the "spiral approach." This guide contains the Language of Strings and Arrows; Geometry and Measurement; and Probability and Statistics. The languages are intended to provide a precise way to communicate ideas about classification and relations. The String Game is again presented to encourage critical thinking. Measurement and parallelism are included in the Geometry section of the guide. Random number stories and probability games are a part of the section on Probability and Statistics. (Author/SA)

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CSMP Mathematics for the Intermediate Grades

Part IV

Teacher's Guide

The Languages of Strings and Arrows
Geometry and Measurement
Probability and Statistics

Comprehensive School Mathematics Program
CEMREL, Inc.

Experimental Version

5-38301

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Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades

Part IV

The Languages of Strings and Arrows

CEMREL, Inc.

TABLE OF CONTENTS

SUGGESTED CHRONOLOGICAL ORDER OF L-LESSONS v

INTRODUCTION vii
Classification: The Language of Strings vii
Relations: The Language of Arrows ix

CHAPTER 1: STRING GAMES AND LOGICAL THINKING 1
1.1 String Game Analysis #1 3
1.2 Counting Problems #1 12
1.3 String Game Analysis #2 20
1.4 Counting Problems #2 34
1.5 String Game Analysis #3 45
Worksheets L01, L02 56

CHAPTER 2: IMAGES 57
2.1 Images #1 59
Worksheet L03 76
2.2 Images #2 77
Worksheets L04, L05, L06 92
2.3 Guess My Rule #1 93
Worksheets L07, L08 100
2.4 Images #3 101
Worksheets L09, L10, L11 111
2.5 Guess My Rule #2 112
Worksheets L12, L13, L14, L15, L16 119

CHAPTER 3: DISTANCE 121
3.1 Map Distance #1 123
Worksheets L18, L19 130
3.2 Map Distance #2 131
Worksheet L20 144

3.3 Parental Distance 145
3.4 Prime Factor Distance #1 155
Worksheets L21, L22 167
3.5 Prime Factor Distance #2 168

APPENDIX: The String Game 181

Equipment 181

Preparation for the Game 184

Object of the Game 185

Rules of the Game 185

Version A: The String Game with A-Blocks 186

Version B: The String Game with A-Blocks, using "NOT" Cards 189

Version C: The String Game with Numbers: Multiples and Order 191

Version D: The String Game with Numbers: Multiples, Divisors, and Order 193

Version E: The String Game with Special Scoring Rules ... 196

6

SUGGESTED CHRONOLOGICAL ORDER OF L - LESSONS

Lesson One	L1.1	String Game Analysis #1	3
Lesson Two	L1.2	Counting Problems #1	12
Lesson Three	L2.1	Images #1	59
Lesson Four	L3.1	Map Distance #1	123
Lesson Five	L3.2	Map Distance #2	131
Lesson Six	L1.3	String Game Analysis #2	20
Lesson Seven	L2.2	Images #2	77
Lesson Eight	L3.3	Parental Distance	145
Lesson Nine	L2.3	Guess My Rule #1	93
Lesson Ten	L1.4	Counting Problems #2	34
Lesson Eleven	L2.4	Images #3	101
Lesson Twelve	L2.5	Guess My Rule #2	112
Lesson Thirteen	L1.5	String Game Analysis #3	45
Lesson Fourteen	L3.4	Prime Factor Distance #1	155
Lesson Fifteen	L3.5	Prime Factor Distance #2	168

THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings is the nonverbal language of sets, that is, collections of objects put together in some way. The language of arrows is the nonverbal language of relations among objects. These languages permeate the CSMP content strands and are used, separately or together, to present an unending variety of numerical and nonnumerical situations.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says: sets. If the mathematics of sets has any place in an elementary school mathematics curriculum, it is precisely because it can help students to understand and use the ideas of classification.

The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (s is not in S). We represent this simple act of sorting -- "in" or "out" -- neatly by using string pictures to illustrate "in" and "out" in a dramatic way. Objects to be sorted are represented by dots and the sets into which they are sorted are represented by drawing "strings" around dots. A dot inside (the region delineated by) a set's string is for an object in the set and a dot outside a set's string is for an object not in the set. (See for example the storybook Where's My Nose?.)

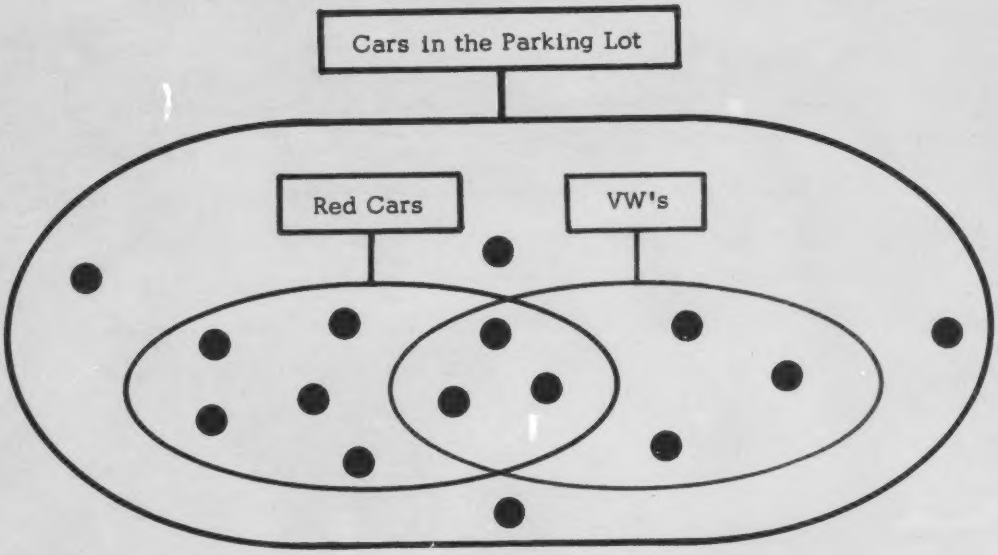
This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The abilities to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity, in general, and for the pursuit and understanding of mathematics, in particular. Ordinarily, we are not expected to deal with such problems until our command of our natural language is fully developed and until we face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires a special mode of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. If frees young minds to think logically and creatively about classes and to report their thinking long before they have extensive verbal skills.

In this strand we present situations and ask carefully phrased questions to continue to advance skills in classification, always remembering that the skills grow out of such experiences. To be able to draw strings and dots is not an objective in itself; to develop the mode of thinking involved is the objective. Thus it is important for us to construct the situations carefully. The sets into which we ask students to classify objects must be determined by well-defined attributes; otherwise, the students have the added problem of deciding whether a certain object does or does not have a certain attribute.

For example, consider this situation. A teacher announces that the blue string on the board is for all children in the class with dark hair and the red string is for all children in the class who are tall. The teacher invites John, a student in the class, to locate a dot for himself in the picture. He places a dot inside the blue string and outside the red string and labels it "J". There is a flurry of mixed agreement and disagreement in the classroom. John has brown hair and he thinks it is dark, but some say it is really light. John does not think he is tall, but someone insists that he is taller than most of the other children in the class. As a result, most of the energy in the class is directed toward making a decision about what attributes John has in this situation rather than how to classify John relative to those attributes. More care in choice of attributes would have avoided such distraction and confusion. For example, if the blue string were for all children in the class whose birthdays are in January and the red string for all the children who have a sister, John's choice of a dot for himself would be criticized strictly on its placement (i.e., his classification of himself) and not on his decision as to what attributes he has.

For this reason we make extensive use of A-Blocks (Attribute Blocks) and sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, and so on. They know whether or not a given number is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to count the objects that have a certain attribute. Suppose for example, in a parking lot there are eight red cars and six Volkswagons. If four cars in the lot are neither red nor Volkswagons, can there be 18 cars in the lot? Suppose we count the cars in the parking lot and find there are exactly 15. Is this possible? A string picture immediately settles this apparent paradox.



There are three red Volkswagons.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in nonmathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphical representation of a relation, somewhat the same way that a blueprint is a handy graphical representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way -- usually more conveniently than the same information could be presented in words. (See for example the lessons on the storybook A Valentine Mystery.)

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read" -- and also draw -- an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical, motor problems of writing or reading about the ideas in conventional systems. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Another educational bonus occurs when arrow diagrams are used imaginatively. A story or problem may be captured by an arrow diagram, and at the same time, the resulting arrow diagram may suggest other, similar situations. This allows students to call upon previous experience or to expand their creativity.

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. We resist making the ability to draw prescribed arrows an objective in itself, just as we resist viewing an arrow diagram as just another format for drill problems in arithmetic.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze.

By this time your students should be very familiar and quite comfortable with both the language of strings and the language of arrows. The content of this strand for the present semester builds in a variety of ways on this fluency with the languages. Each of the three chapters included in this strand present situations quite distinct from the other.

Chapter 1 extends the String Games and logical thinking lessons from Part III (Chapter 2 of the L strand). Three of the lessons continue a systematic analysis of a mathematically rich game (The String Game) which has become familiar to the students. Two lessons extend logical thinking lessons to include systematic methods for solving certain counting problems with sets.

Chapter 2 continues the work with relations, functions, and composition from previous parts of the IG program. Here the internal structure of the languages of strings and arrows is exploited to explore situations involving images of sets under certain relations. The situations in this chapter lend support to and continue work begun in both the World of Numbers strand and the Geometry and Measurement strand.

Chapter 3 presents the notion of distance in several seemingly unrelated situations. The language of arrows (and cords) provides the tool for studying a general distance notion and for discovering some properties of a distance function.

1

STRING GAMES and LOGICAL THINKING

The Language of Strings is used as a vehicle to present various situations involving sets which stimulate logical thinking. The situations presented in the lessons of this chapter are similar to those experienced earlier in the CSMP curriculum, and special attention is given to situations involving the already very familiar String Games. Continuing the experiences in the L strand of CSMP Mathematics for the Intermediate Grades, Part III, the String Game situations involve the students in thinking critically about the relations "is a divisor of", "is a multiple of", "is larger than", etc., as well as challenge them to think logically about the relationships between various sets and to give persuasive arguments. Two lessons of the chapter involve the students in some counting problems with sets and subsets. Explicit attention is paid to statements involving precise terms such as: at least, at most, exactly, etc. The situations are rich, they encourage deductive reasoning on the part of students, and with the aid of string pictures, they draw upon the students' numerical knowledge to solve problems.

CAPSULE LESSON SUMMARIES

Section 1.1: Determine two string labels from three clues one at a time. Play the String Game with two strings. (Optional) Analyze the information obtained from the starting clues of a String Game with three strings and then play the game to determine the string labels.

Section 1.2: Present counting problems in a variety of settings where information about a set and two intersecting subsets is given.

Section 1.3: In a two strings picture, determine the string labels from three starting clues either individually, given the clues all at once, or collectively, given the clues one at a time. Analyze the information obtained from a series of four clues in a string picture with three strings and then discuss which of the remaining pieces would determine the string labels. Play the String Game.

Section 1.4: Given a string picture, determine whether statements about the picture are true, false, or you can't tell. Include statements involving "exactly", "at most", and "at least". Given a statement, make additions to a string picture so that the statement is true. Extend the experiences in Section 1.2 to consider a set and three intersecting subsets.

Section 1.5: Play the String Game with three strings or simply place the game pieces in the picture with the string labels face-up. Present a three strings picture where one string is for prime numbers and the other two strings are each for multiples of an unknown number. Discover the unknown numbers by the placement of various numbers in the string picture.
(Worksheets L01, L02)

1.1 String Game Analysis #1

CAPSULE LESSON SUMMARY

Determine two string labels from three clues given one at a time. Play the String Game with two strings. (Optional) Analyze the information obtained from the starting clues of a String Game with three strings and then play the game to determine the string labels.

MATERIALS

Teacher: Colored chalk; Numerical String Game kit; Demonstration Analysis Sheets (two or five)

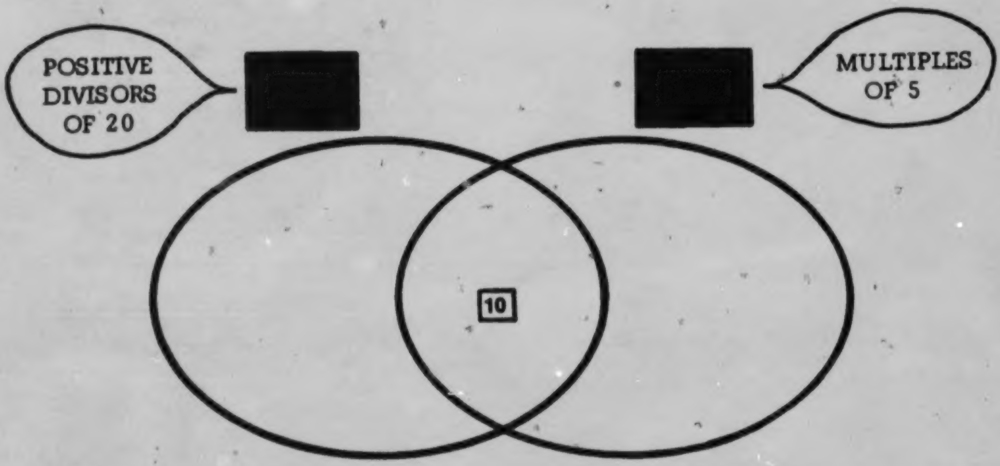
Student: String Game Analysis Pad (two or three strings); pencil

DESCRIPTION OF LESSON

Exercise 1 is a warm-up activity and should move quickly since it is a review. Exercise 3 is optional, but should be appropriate for most average to advanced classes.

Exercise 1

Before the lesson begins, set up your board for the String Game as illustrated below. The bubbles show what is on the hidden tags.



MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

L1

T: This is a starting situation for a String Game. Here is the list of possibilities for the red string (point to the analysis sheet at the left) and here is the list for the blue string (point to the analysis sheet at the right). The first clue is that the number 10 is in the center region; 10 is inside both strings. What information does this clue give us about the strings?

S: The red string cannot be for multiples of 3, because 10 is not a multiple of 3.

T: Could the blue string be for multiples of 3?

S: No, because 10 is also inside the blue string.

Use a crayon or heavy felt-tip pen to cross out this possibility from both lists.

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^-10$
SMALLER THAN 50	SMALLER THAN $^-10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^-10$
SMALLER THAN 50	SMALLER THAN $^-10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

Continue to encourage the class to eliminate as many labels as possible using this clue. Students should realize that a number in the center region gives the same information about both strings. Cross out a label only after adequate explanation has been given. For example,

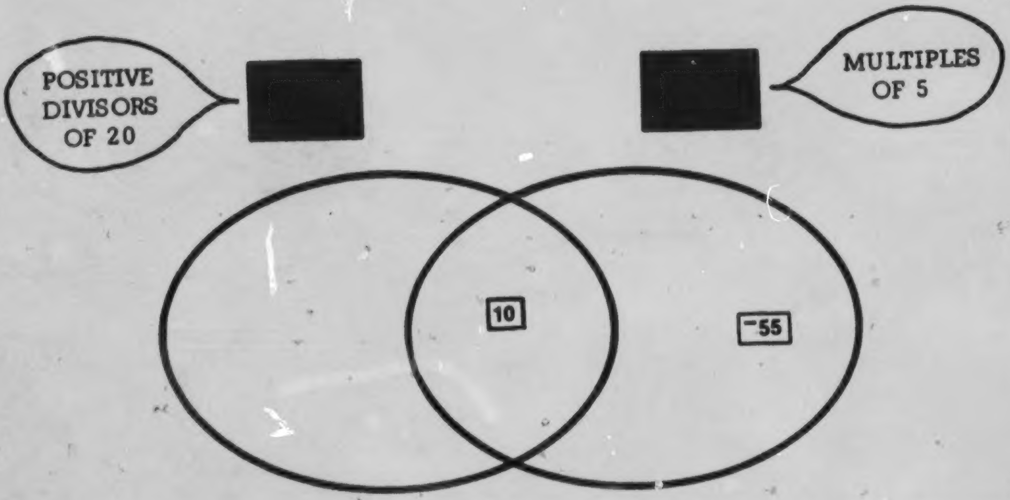
S: The red (or blue) string can not be for larger than 50, since 10 is smaller than 50.

When all the information from the clue has been discussed, there will be six possibilities remaining for each string.

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
MULTIPLES OF 10	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN ⁻10
SMALLER THAN 50	~~SMALLER THAN ⁻10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
MULTIPLES OF 10	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN ⁻10
SMALLER THAN 50	~~SMALLER THAN ⁻10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: This is the second clue. ⁻55 is in the blue string but not in the red string. (Put ⁻55 on the board.) What new information does this give us?



Cross off labels as suggested and correctly justified by the students. Lead your class to realize that with this type of clue: if the blue string cannot have a particular label, that the red string can have it, and vice versa. For example,

S: The blue string cannot be for numbers larger than ⁻10, because ⁻55 is smaller than ⁻10.

T: Could the red string be for numbers larger than ⁻10?

S: ⁻55 is not in the red string. Yes, the red string could be for numbers larger than ⁻10.

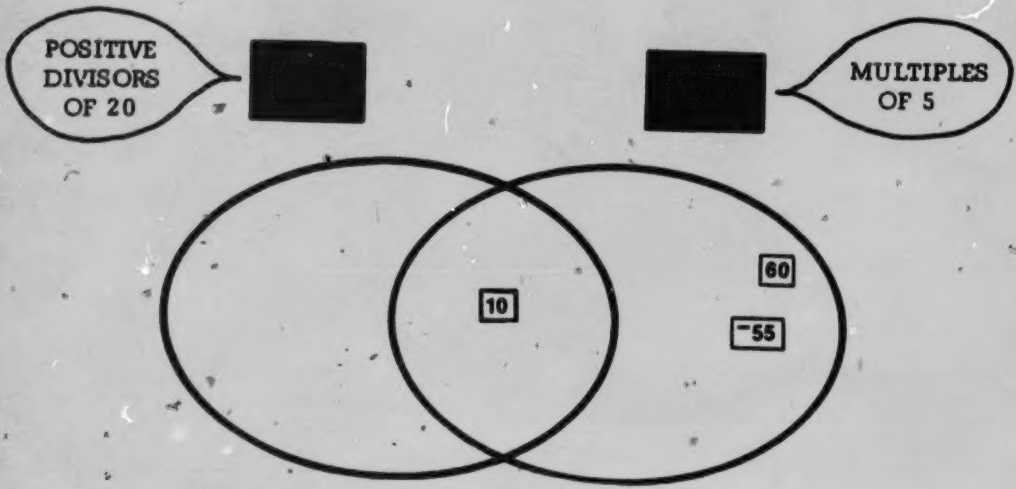
L 1

When all the information from this clue has been discussed, there should be these possibilities remaining.

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN $^{-}10$
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
SMALLER THAN 50	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

T: There are four possibilities remaining for the red string and two for the blue. The third clue is that 60 is in the blue string and not in the red string. (Put $\boxed{60}$ on the board.)



Let the students eliminate labels from both lists until the red and blue strings have been determined.

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

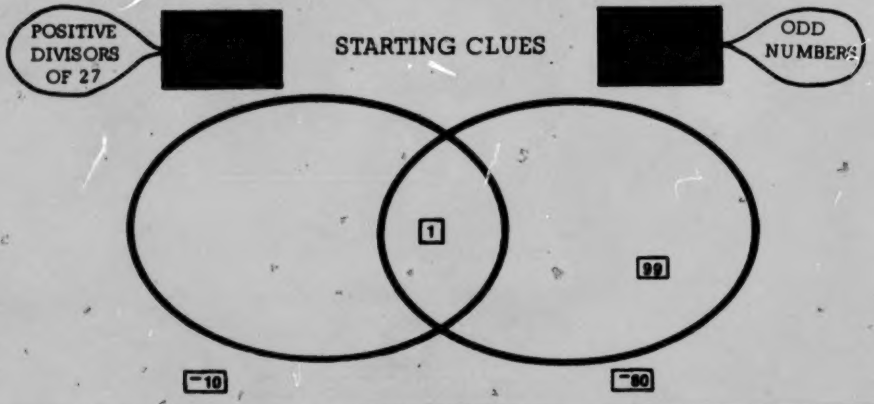
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN $^{-}10$~~
~~SMALLER THAN 50~~	~~SMALLER THAN $^{-}10$~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

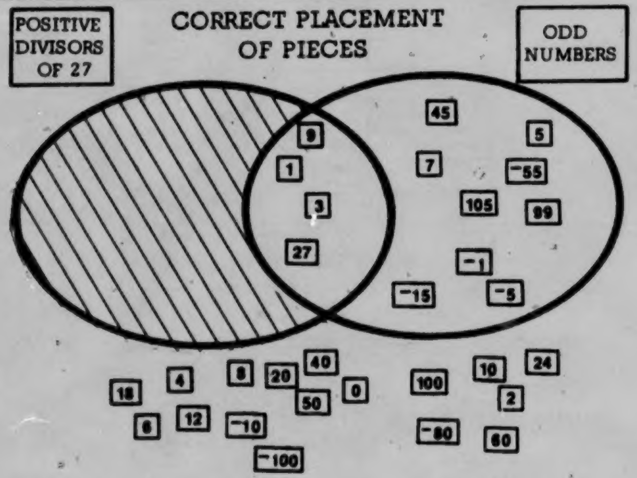
Exercise 2

Play Version D (variation #1 with two strings) of the String Game with numbers. (See the Appendix on the String Game for complete details on how to play the different versions of the game.) Give each student a list of the string labels from the individual String Game Analysis Pad.

T: Now we are going to play the String Game. Each of you has a list of the possible labels for the red string and the blue string. Numbers placed in the picture will serve as clues, allowing you to cross out labels from your lists.

Allow time in between turns for students to cross out labels from their lists. One game is suggested here in which there is an empty region; however, you should decide what kind of game would be appropriate for your students.

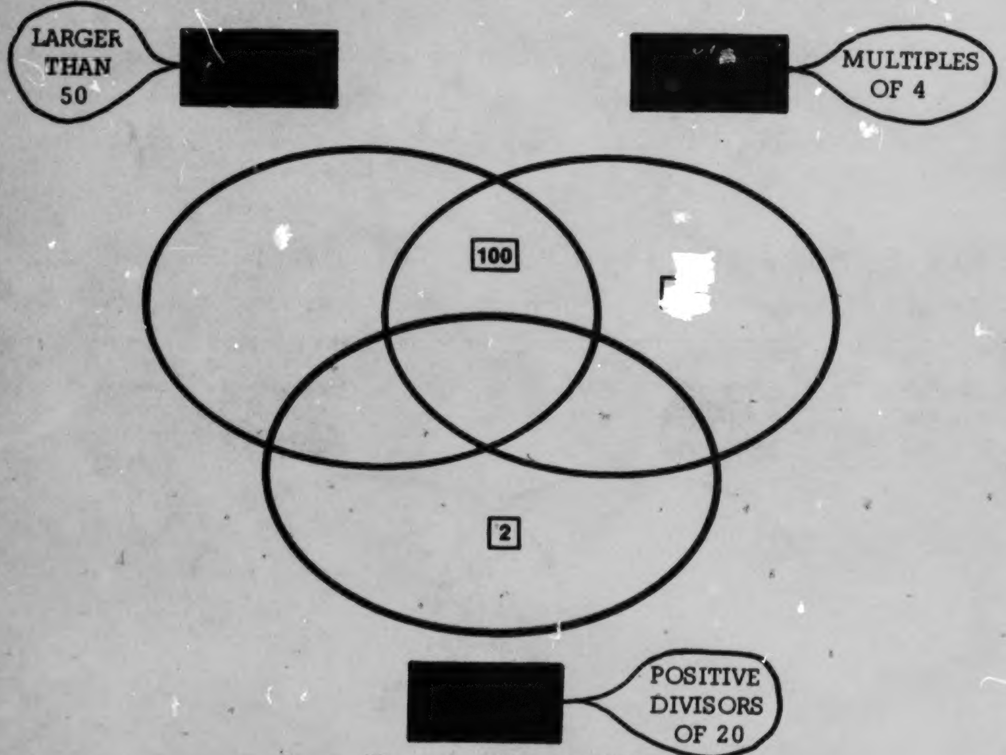

POSITIVE DIVISORS OF 27
STARTING CLUES
ODD NUMBERS
1
99
-10
-80


POSITIVE DIVISORS OF 27
CORRECT PLACEMENT OF PIECES
ODD NUMBERS
9
1
3
27
45
5
7
-55
105
99
-1
-15
-5
18
4
8
20
40
0
50
100
10
24
2
6
12
-10
-100
-80
60

L 1

Exercise 3

Distribute sheets from the individual String Game Analysis Pad for three strings to the students. Set up your board as illustrated below. Tape a three string demonstration analysis sheet on the board.



RED	BLUE	GREEN
MULTIPLES OF 2	MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	LARGER THAN 50	LARGER THAN 50
SMALLER THAN 50	SMALLER THAN 50	SMALLER THAN 50
LARGER THAN $^{-}10$	LARGER THAN $^{-}10$	LARGER THAN $^{-}10$
SMALLER THAN $^{-}10$	SMALLER THAN $^{-}10$	SMALLER THAN $^{-}10$
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

T: Let's look first at the number clues. What information do they give us about the strings?

Collectively, let students suggest labels to cross out on each list. Ask for adequate explanation before crossing out a label. Encourage analysis of more than one string at a time. For example,

S: The green string cannot be for multiples of 2, because 100 and 12 are not in the green string.

T: Can the red or the blue strings be for multiples of 2?

S: No, because 2 is neither in the red string nor the blue string.

T: So we can cross out multiples of 2 for all three strings.

Another example,

S: Cross out positive divisors of 12 for all three strings.

T: Convince the class.

S: 12 is not in the green or in the red string, and 12 is a positive divisor of 12. The red string has 100 in it, and 100 is not a divisor of 12.

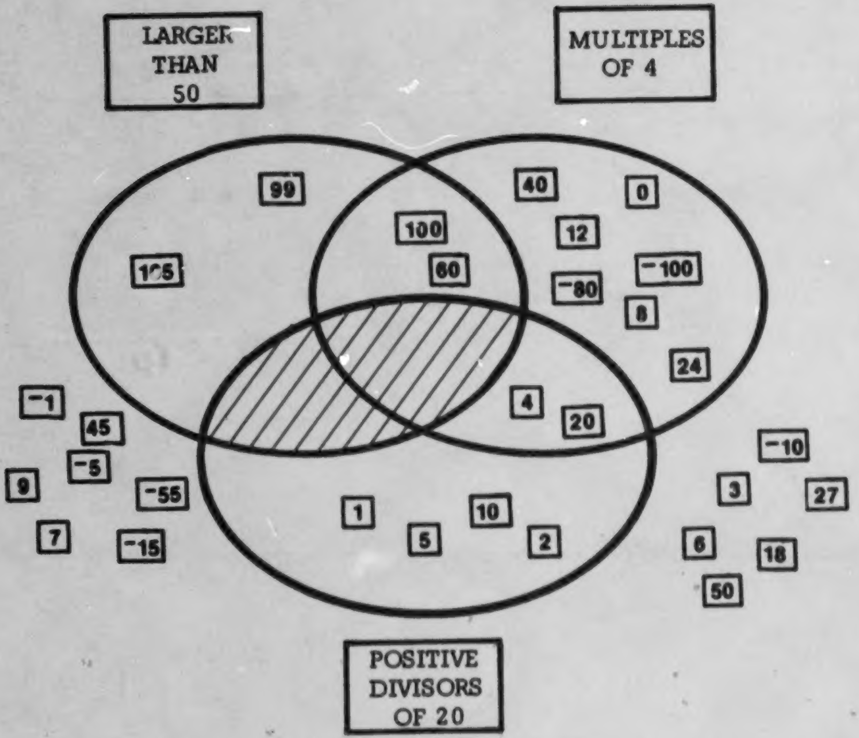
L 1

When all the information from these four clues has been discussed, your class should find these possibilities remaining.

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4	~~MULTIPLES OF 4~~
MULTIPLES OF 5	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS
LARGER THAN 50	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

At this point in the lesson let the students play the String Game in the usual way using the teams from Exercise 2. If there is not enough time to place all the pieces, ask the students to announce what the strings are for when they have been determined by played pieces. The next illustration shows correct placement of all the pieces.

L 1


LARGER THAN 50
MULTIPLES OF 4
POSITIVE DIVISORS OF 20
99
125
100
60
40
0
12
⁻100
⁻80
8
24
4
20
1
10
5
2
⁻1
45
⁻5
9
⁻55
7
⁻15
⁻10
3
27
6
18
50

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	[✓ MULTIPLES OF 4]	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
[✓ LARGER THAN 50]	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	[✓ POSITIVE DIVISORS OF 20]
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

23

L 2

1.2 Counting Problems #1

CAPSULE LESSON SUMMARY

Present counting problems in a variety of settings where information about a set and two intersecting subsets is given.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board as you say each statement.

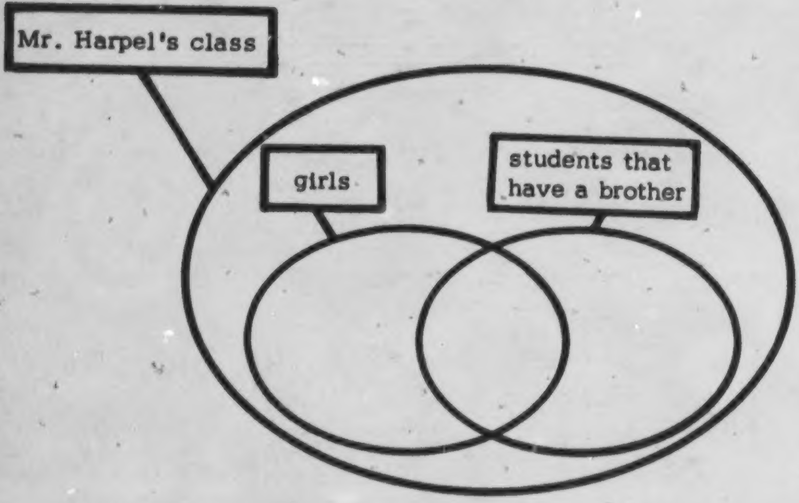
(1) There are 30 students in Mr. Harpel's class. (Any teacher's name would be appropriate.)

(2) There are 16 girls in the class.

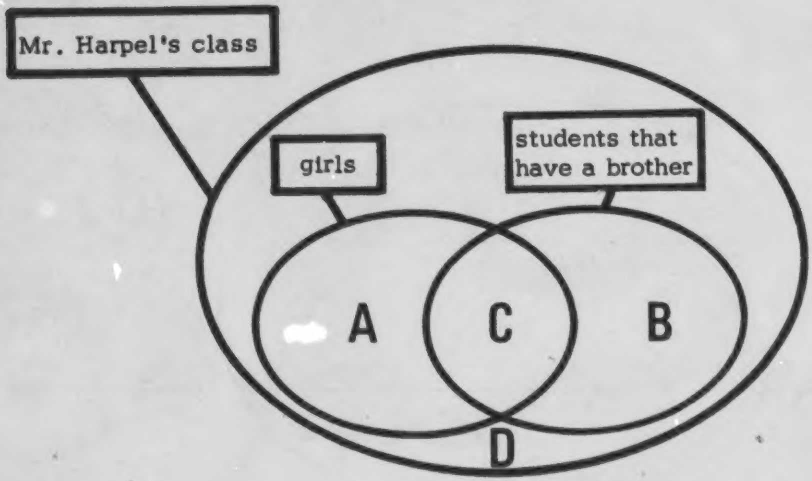
(3) 7 girls in the class do not have brothers.

(4) 12 students in the class have brothers.

Draw this string picture on the board starting with the green string.



Considering the information given, let the class decide whether each statement tells the number of dots in a particular string or whether it gives the number of dots in one of the regions. The regions are lettered in the next picture.

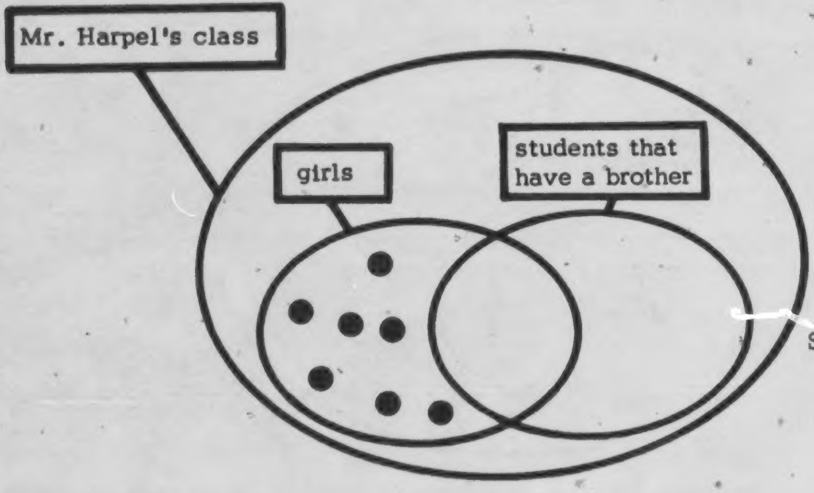


Mr. Harpel's class
girls
students that have a brother
A
C
B
D

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

With the given information, the class should conclude:

- There are 30 dots inside the green string. Write "30" on the board away from the string picture.
- There are 16 dots inside the red string. Write "16" under "30".
- There are 12 dots inside the blue string. Write "12" under the other two numerals.
- Region A has 7 dots. Draw them in the picture.



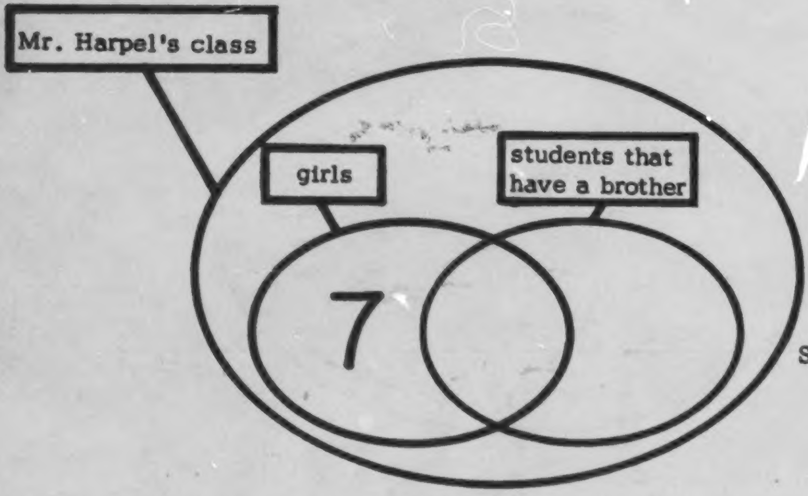
Mr. Harpel's class
girls
students that have a brother

STRING TOTALS
30
16
12

L 2

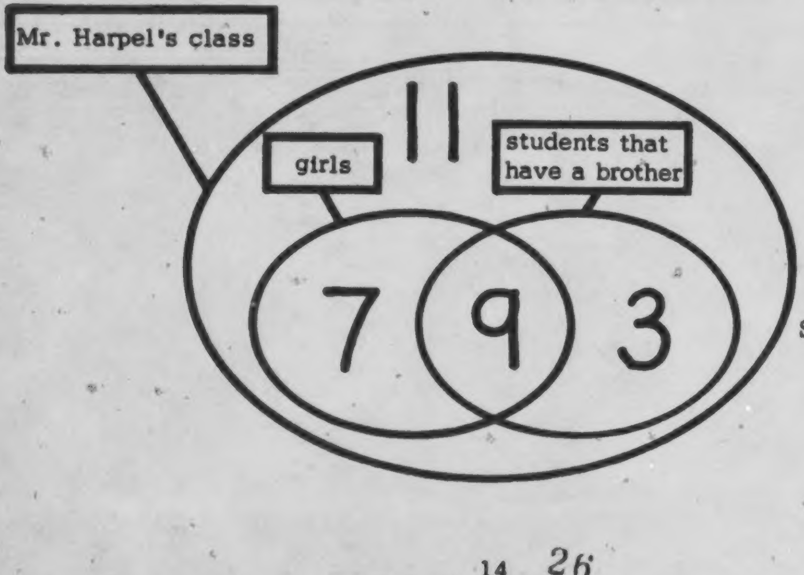
T: We have 30 dots to draw in the green string. Instead of drawing all of the dots, I am going to use a short-hand way of indicating how many dots are in a region. Watch what I do on the board.

Erase the dots and write "7" in Region A.


Mr. Harpel's class
girls
students that have a brother
7

STRING TOTALS
30
16
12

Let the students analyze the situation. The information is sufficient to determine the number of dots in each region. You may want to ask students to draw the picture on a piece of paper and to record in the picture the number of dots in each region. Then you would go over the results collectively. Otherwise, decide the number of dots in each region as a full group activity. In either case, ask students to explain their answers.


Mr. Harpel's class
11
girls
students that have a brother
7
9
3

STRING TOTALS
30
16
12

Afterward, ask these questions which require the students to interpret the regions in the string diagram.

(1) How many girls in the class have brothers? (9)

(2) How many boys in the class do not have brothers? (11)

Exercise 2

Present the situation which follows in a manner similar to that in Exercise 1.

Initial Information:

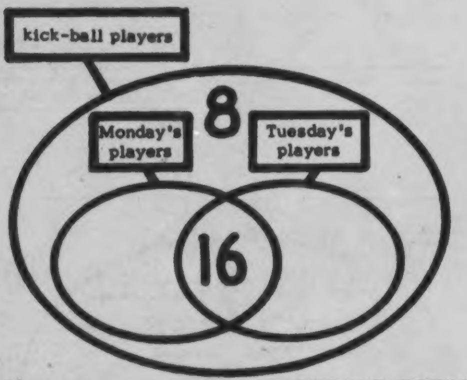
(1) There are 40 students in the school who like to play kick-ball after school hours.

(2) 22 of them play kick-ball on Mondays.

(3) 16 of them play kick-ball on Mondays and on Tuesdays.

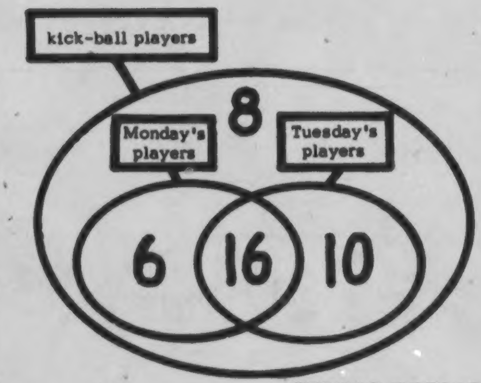
(4) 8 students do not play kick-ball on Mondays nor on Tuesdays.

INITIAL INFORMATION


kick-ball players
8
Monday's players
Tuesday's players
16

STRING TOTALS
40
22

AFTER FURTHER ANALYSIS


kick-ball players
8
Monday's players
Tuesday's players
6
16
10

STRING TOTALS
40
22
26

Concluding Questions:

(1) How many students play kick-ball on Mondays but not on Tuesdays? (6)

(2) How many students play kick-ball on Tuesdays? (26).

L 2

NOTE: If these problems are very difficult for most of your students, this might be a good place to stop in the lesson and to play the String Game with Numbers. In this case, Exercises 3 and 4 would be postponed until the next lesson of this strand.

Exercise 3

Present the situation which follows in a manner similar to that in Exercise 1. This situation requires more analysis than either of the previous exercises. Encourage students to experiment with the problem — to try some numbers and if they do not satisfy the given information, to try some other numbers until a solution is discovered.

Initial Information:

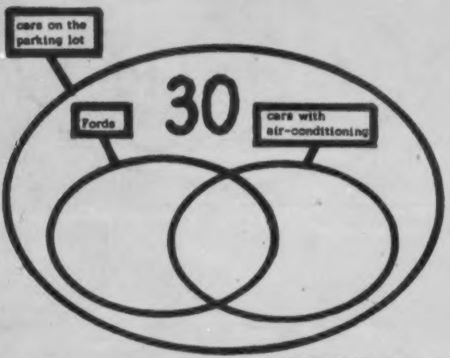
(1) There are 100 cars on the parking lot.

(2) 30 cars on the lot are not Fords and do not have air-conditioning.

(3) 50 cars on the lot are Fords.

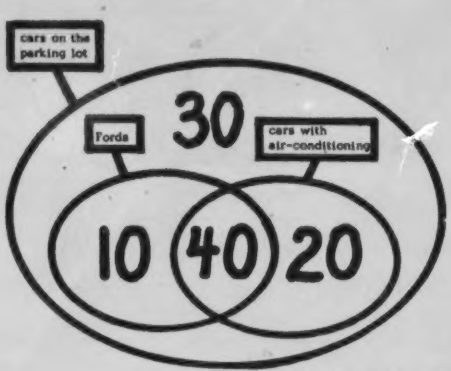
(4) 60 cars on the lot have air-conditioning.

INITIAL INFORMATION


cars on the parking lot
Fords
30
cars with air-conditioning

STRING TOTALS
100
50
60

AFTER FURTHER ANALYSIS


cars on the parking lot
Fords
30
cars with air-conditioning
10
40
20

STRING TOTALS
100
50
60

Concluding Questions:

(1) How many air-conditioned cars on the lot are not Fords? (20)
(2) How many Fords on the lot do not have air-conditioning? (10)
(3) How many air-conditioned Fords are on the lot? (40)

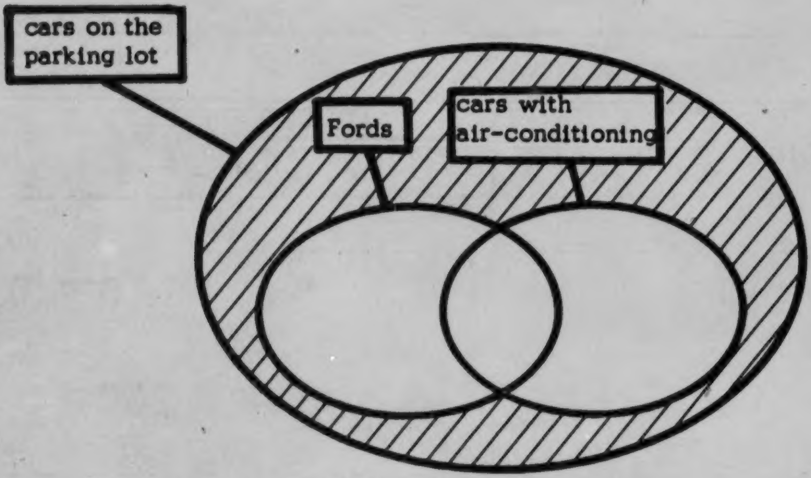
Exercise 4

Erase only the numbers in the string picture from Exercise 3.

T: Suppose all the cars on the parking lot are either Fords or have air-conditioning. How could we indicate this information in the picture?

S: Hatch the region inside the green string but outside the red and the blue strings.

Let a student do the hatching.

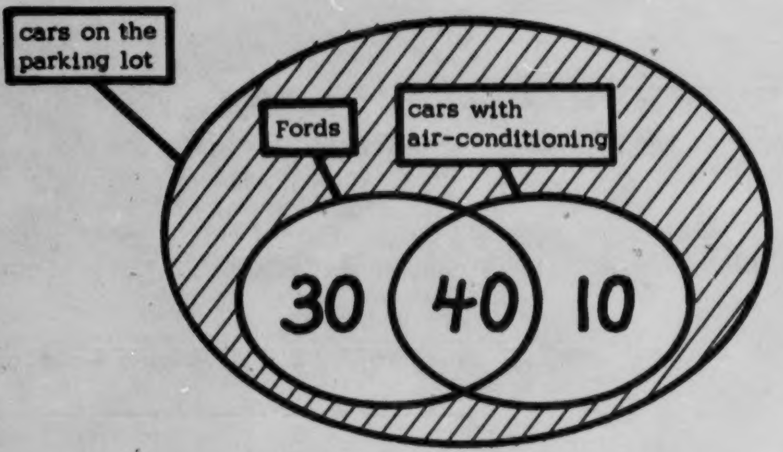


T: This is the other information:

- There are 80 cars on the parking lot. (Write "80" near the picture.)
- There are 70 Fords on the lot. (Write "70" under "80".)
- There are 50 air-conditioned cars on the lot. (Write "50" under the other two numerals.)

L 2

Ask the students to draw the picture on paper and then to write in the picture the number of dots in each region. Again encourage the students to experiment with various combinations of numbers. After a short while, invite someone to put the correct solution on the board.



80
70
50

Repeat this activity several times. Some possibilities for string totals are suggested below. The number of dots in the middle region (the intersection of the strings) is given for each problem.

STRING TOTALS			NUMBER OF DOTS IN THE MIDDLE REGION
120	90	40	(90 + 40) - 120 = 10
200	75	130	(75 + 130) - 200 = 5
60	50	45	(50 + 45) - 60 = 35
24	12	12	(12 + 12) - 24 = 0

NOTE: In each suggested situation, care was taken in choosing the numbers; that is,

(i) the "green number" is not smaller than either the "red number" or the "blue number"; and

(ii) the "green number" is not larger than the "red number" plus the "blue number".

If you use other combinations of numbers for the string totals, they too will need to meet these conditions in order for there to be a solution to the problem.

If your students have not already commented on how to find the number of dots in the middle region, ask:

T: Is there any way to know how many dots will be in the middle region (point to the intersection of the strings)?

To answer this question, a student might refer to an example, perhaps the one with string totals 60, 50, and 45.

S: 50 + 45 = 95, which is 35 more than 60. Those 35 dots must have been counted twice, so they go in the middle region.

If there is time remaining and there is student interest, ask a student to suggest the number of dots in each string; that is, the "green number", the "red number", and the "blue number". Then let the class solve the problem. If the numbers suggested do not yield a solution, let the class discover this rather than correcting the student who is giving the data.

L 6

1.3 String Game Analysis #2

CAPSULE LESSON SUMMARY

In a two string picture, determine the string labels from three starting clues either individually, given the clues all at once, or collectively, given clues one at a time. Analyze the information obtained from a series of four clues in a string picture with three strings and then discuss which of the remaining pieces would determine the string labels. Play the String Game.

MATERIALS

Teacher: Numerical String Game kit; colored chalk; Demonstration Analysis Sheets; magnetic board (optional)
Student : String Game Analysis Pad (three strings)

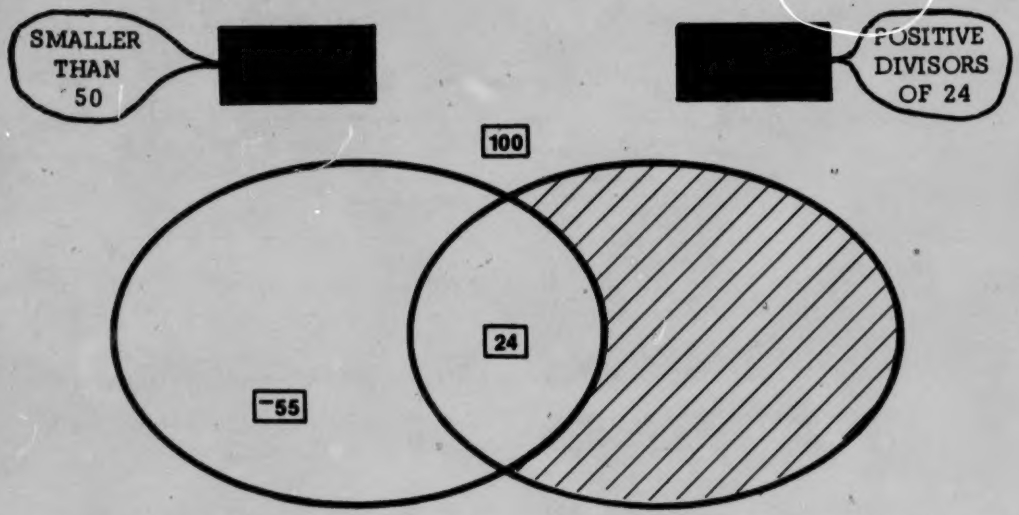
DESCRIPTION OF LESSON

NOTE: To allow for a wide range in students' abilities, several options have been built into this lesson. With average to advanced classes, begin with Exercise 1A and proceed to Exercise 2A. If most of your class consists of weaker students, begin with Exercise 1B and proceed to Exercise 2B.

Exercise 1A

Prepare your board for the String Game as in the next illustration. Tape two demonstration analysis sheets to the board near the strings. Distribute individual sheets from the String Game Analysis Pad to the students.

NOTE: The sheets from the String Game Analysis Pad (three strings) are easily adapted for use in situations involving only two strings; simply ignore the third column headed "GREEN".



MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^-10$
SMALLER THAN 50	SMALLER THAN $^-10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^-10$
SMALLER THAN 50	SMALLER THAN $^-10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

Remind the class that one of the regions is hatched because it is empty.

T: With these three number clues and the hatching, you can determine the strings. Use your analysis sheets to help you find the labels.

Give the students sufficient time to analyze the situation. Walk around the room and observe the students' conclusions. Most students will be able to determine the red string and to narrow the blue string down to two possibilities: multiples of 3 and positive divisors of 24. Begin a discussion when many students have reached this point.

T: Almost everyone found that the red string is for numbers smaller than 50.

L 6

Circle this label on the demonstration analysis sheet for the red string.

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
[✓ SMALLER THAN 50]	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

T: Some of you found that the blue string could be for the multiples of 3 or for the positive divisors of 24. How can we decide which it is?

S: There are multiples of 3 which are larger than 50, but that region in the picture is hatched. So the blue string cannot be for the multiples of 3.

T: Who can give a multiple of 3 which is larger than 50?

S: 60.

T: Is 60 smaller than 50? (No)
Is 60 a multiple of 3? (Yes)
Where would 60 be in the picture? (The hatched region)
But it can't go there. What are some other numbers which would have no place in the picture if the blue string were for the multiples of 3? (105; 99)

Conclude that the blue string is for the positive divisors of 24 and circle it on the demonstration analysis sheet.

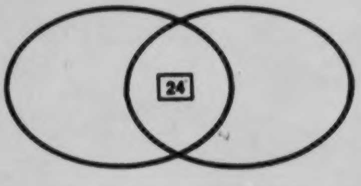
MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
[✓ SMALLER THAN 50]	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
[✓ POSITIVE DIVISORS OF 24]	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN $^{-}10$
SMALLER THAN 50	SMALLER THAN $^{-}10$	ODD NUMBERS	POSITIVE PRIME NUMBERS

Exercise 1B

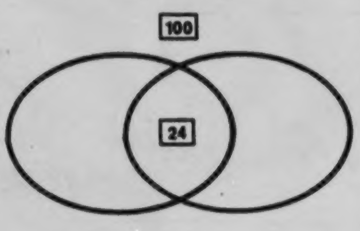
Use the following situation and sequence of clues (present them one at a time) to determine the string labels. Collectively discuss and cross out labels from the lists after each number has been placed in the picture. Discussion of the last clue, hatching one of the regions, is included in the description of Version A of this exercise.

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN ⁻10
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~



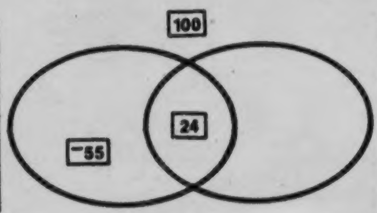
MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	~~MULTIPLES OF 6~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	LARGER THAN ⁻10
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN ⁻10~~
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~



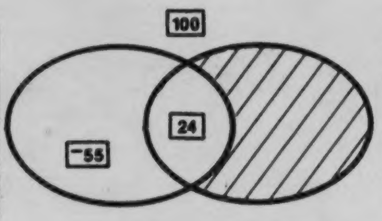
~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 6~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN ⁻10~~
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~



~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 6~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
SMALLER THAN 50	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

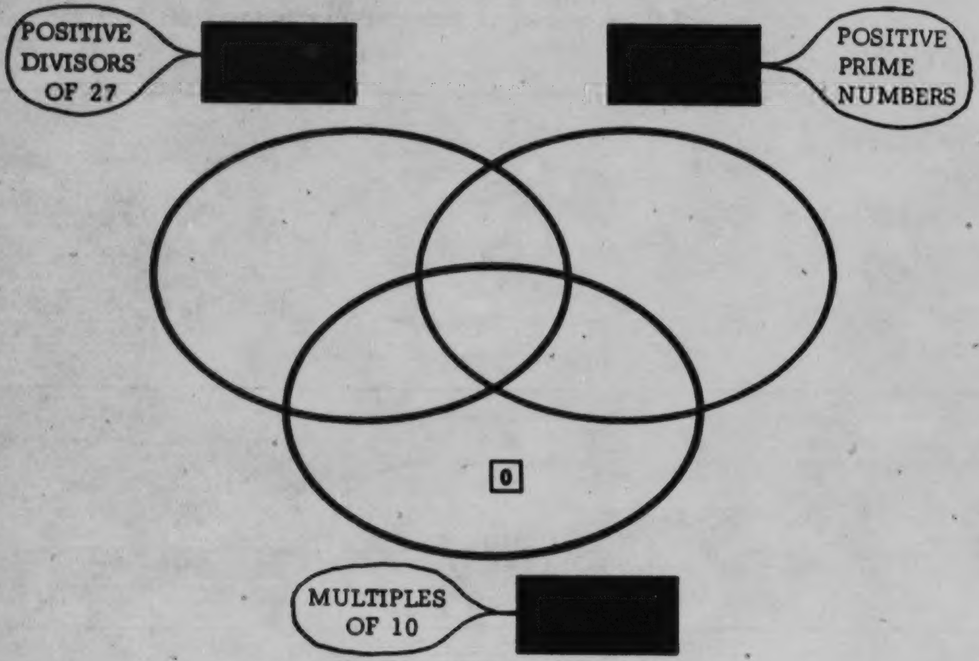


~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 6~~
~~MULTIPLES OF 10~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 16~~	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~	~~LARGER THAN 50~~	~~LARGER THAN 10~~
~~SMALLER THAN 50~~	~~SMALLER THAN 10~~	~~ODD NUMBERS~~	~~POSITIVE PRIME NUMBERS~~

L 6

Exercise 2A

Distribute sheets from the individual String Game Analysis Pad for three strings to the students. Set up your board as illustrated below. Tape a three string demonstration analysis sheet on the board.



RED	BLUE	GREEN
MULTIPLES OF 2	MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	LARGER THAN 50	LARGER THAN 50
SMALLER THAN 50	SMALLER THAN 50	SMALLER THAN 50
LARGER THAN ⁻10	LARGER THAN ⁻10	LARGER THAN ⁻10
SMALLER THAN ⁻10	SMALLER THAN ⁻10	SMALLER THAN ⁻10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

Place $\boxed{0}$ in the picture as indicated and put the other game pieces on the board at a distance from the string picture. They will be used later in the lesson.

First clue

T: The first clue is that the number 0 is in the green string, but not in the red string nor in the blue string. Use the information from this clue to cross out some of the labels from your lists.

After the students have marked their lists, discuss why certain labels can be crossed out. Cross out these labels on the demonstration analysis sheets. Encourage the class to observe that with this clue the same labels can be crossed out for the red and the blue strings; for example,

S: Cross out "Larger Than ¯10" for the red string.

T: Can we cross out this label for the blue string also?

S: Yes, anything we cross out for the red string can be crossed out for the blue string.

T: Why?

S: There are no number clues in the red string nor in the blue string — it's the same situation for both strings.

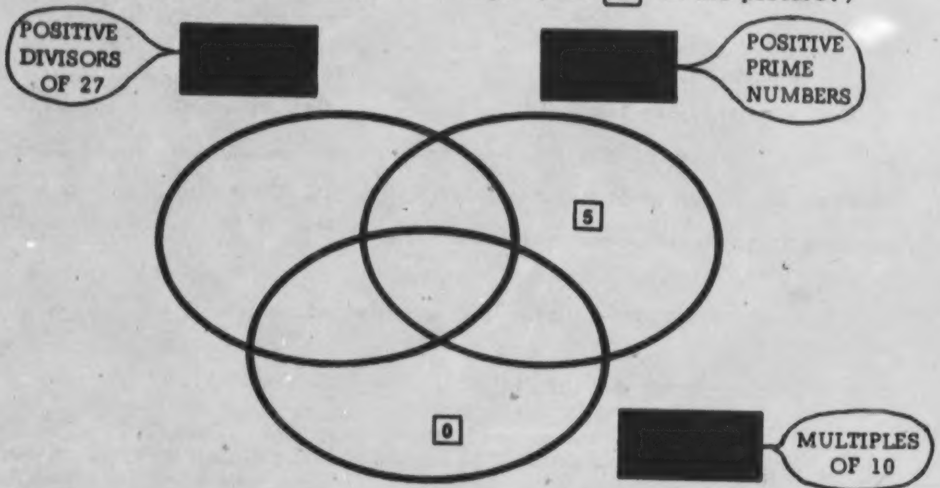
After all the information from this clue has been discussed, these possibilities will be remaining for each string.

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	MULTIPLES OF 3
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
LARGER THAN 50	LARGER THAN 50	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	SMALLER THAN 50
~~LARGER THAN ¯10~~	~~LARGER THAN ¯10~~	LARGER THAN ¯10
SMALLER THAN ¯10	SMALLER THAN ¯10	~~SMALLER THAN ¯10~~
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

L 6

Second clue

T: The second clue is that the number 5 is in the blue string, but not in the red string nor in the green string. (Put $\boxed{5}$ in the picture.)



Consider the strings one at a time and ask students which additional labels can be crossed out of the lists.

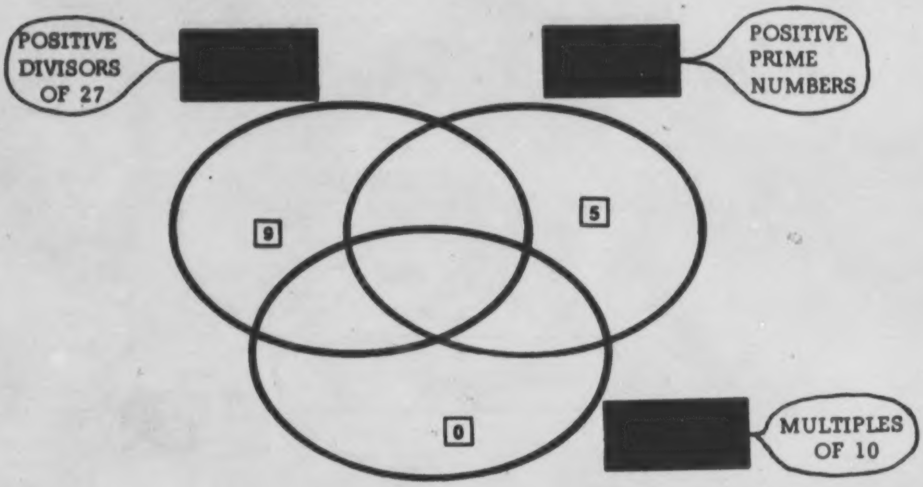
NOTE: A student might suggest that the same labels be crossed out for both the red and the green strings. Such a suggestion is good, however, you should observe that some labels from one of the two strings may already be crossed out as a result of the previous clue. They do not need to be crossed out again.

These will be the remaining labels.

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	MULTIPLES OF 3
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
~~ODD NUMBERS~~	ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
LARGER THAN 50	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~	~~LARGER THAN 10~~
SMALLER THAN 10	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

Third clue

T: The third clue is that the number 9 is in the red string, but not in the blue string or in the green string. (Put $\boxed{9}$ in the picture.) Which additional labels can we cross out?



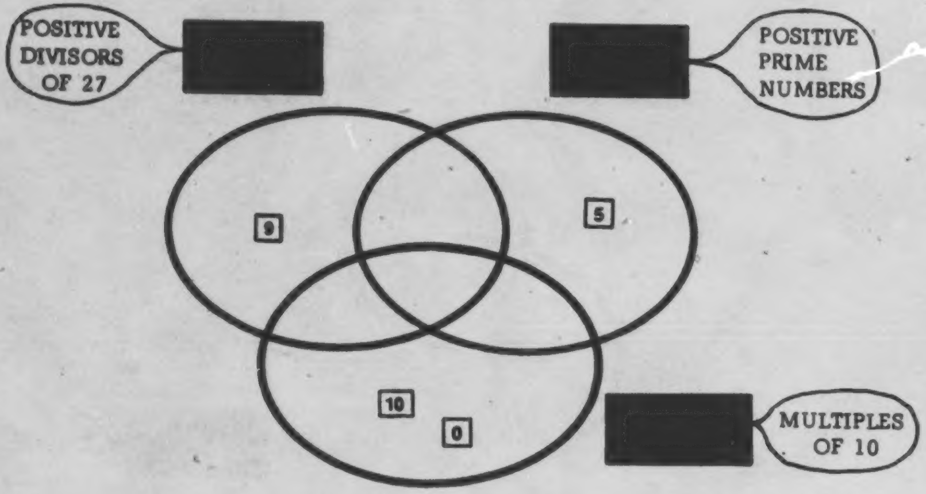
Remaining labels:

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

L 6

Fourth clue

T: The fourth clue is that the number 10 is in the green string, but not in the red string nor in the blue string. (Put $\boxed{10}$ in the picture.) Which additional labels can we cross out?



Remaining labels:

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

At this point, the blue string has been determined and there are two possibilities remaining for the red string and two possibilities for the green string. Depending upon how well your students did with the first part of this exercise, you can proceed in either of two ways.

- Ask the students which game pieces they would like to know the position of in the picture. Place the pieces one at a time, analyzing the information from each correctly placed piece, until the two remaining strings have been determined. (This is by far the easier of the two options.)
- Continue the analysis as in the following dialogue.

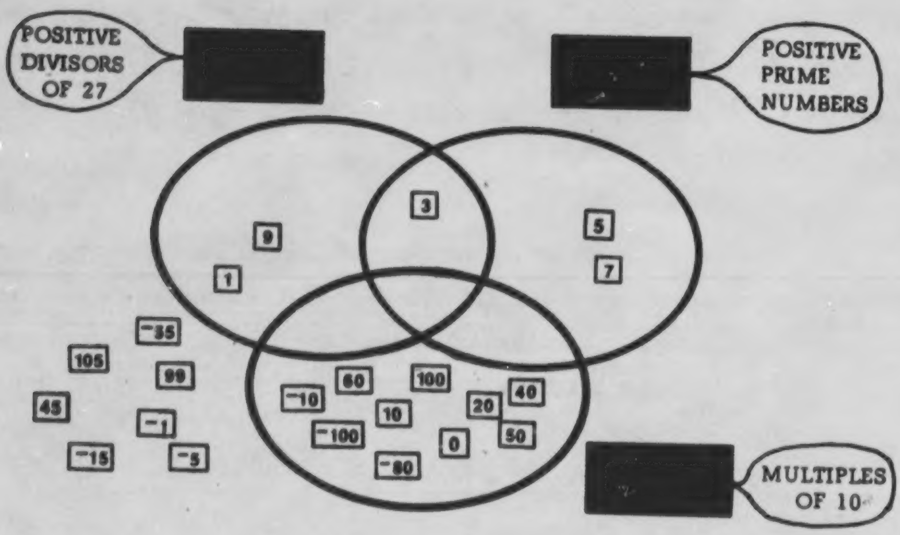
T: We have two possibilities left for the red string: positive divisors of 18 and positive divisors of 27. We also have two possibilities for the green string: multiples of 2 and multiples of 10. The blue string is for the positive prime numbers.

Point to the number pieces on the board which are not in the string picture.

T: There are thirty game pieces. Four of them have been placed in the picture. Of the remaining twenty-six pieces, eighteen of them can be placed correctly in the picture without knowing exactly what the red and the green strings are for. Which numbers can be correctly placed and where do they belong in the picture?

Let the students give numbers, explain where they go in the picture, and place the pieces. Your picture will eventually look like the next illustration.

L 6


POSITIVE DIVISORS OF 27
POSITIVE PRIME NUMBERS
MULTIPLES OF 10
3
9
1
5
7
⁻35
105
99
45
⁻1
⁻15
⁻5
60
100
⁻10
10
⁻100
20
40
50
0
⁻80

These numbers cannot be put in the picture without further information.

8 4 24 6

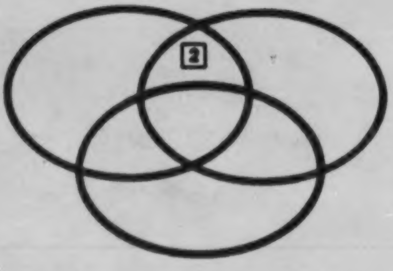
18 12 27 2

T: Among these eight numbers (point to the pieces on the board), there are three which could be used in a single play and would determine both the red and the green strings if we got a "yes" answer. Which three numbers are these? (2, 6, 18)

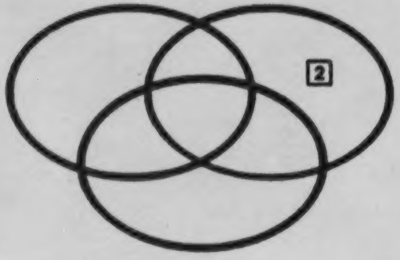
As students make suggestions, indicate to them how many of the three numbers offered are correct (but don't tell them which ones) until all three have been discovered.

T: Look at each of eight regions in this string picture. As I place 2 in each region, tell me if 2 can be there and if so what the red and the green strings are for.

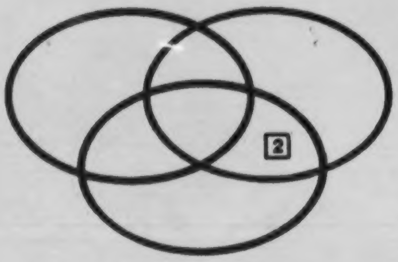
S: We only need to look at the blue string because 2 is a prime number.



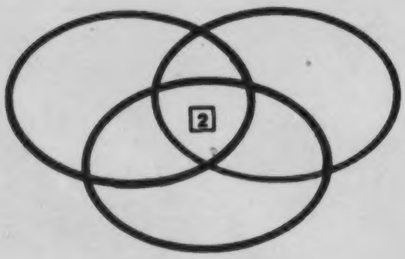
Red : Positive Divisors of 18
Green : Multiples of 10



Red : Positive Divisors of 27
Green : Multiples of 10



Red : Positive Divisors of 27
Green : Multiples of 2



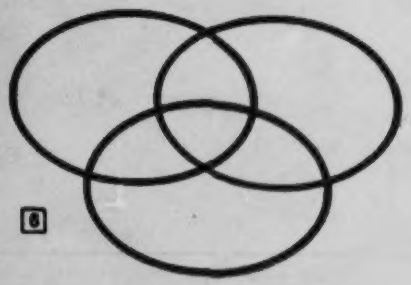
Red : Positive Divisors of 18
Green : Multiples of 2

T: So no matter where (in the blue string) we play 2 if we get a "yes" answer, we will know what the two strings are for.

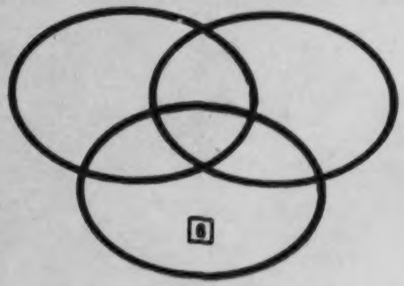
T: 6 and 18 are both positive divisors of 18. Whatever information we obtain from playing 6 will be the same as we would get from playing 18. So let's consider only 6. What do we know about the number 6?

S: It cannot be in the blue string because 6 is not a prime number.

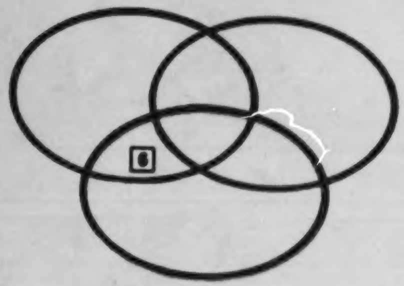
L 6



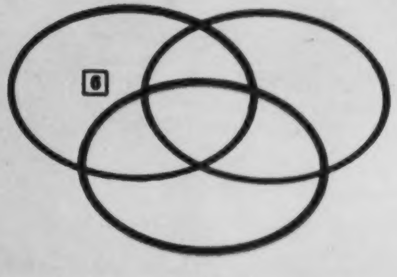
Red : Positive Divisors of 27
Green : Multiples of 10



Red : Positive Divisors of 27
Green : Multiples of 2



Red : Positive Divisors of 18
Green : Multiples of 2



Red : Positive Divisors of 18
Green : Multiples of 10

T: So again, if we play 6 in any of the regions where it could be, a "yes" answer will tell us what the red and green strings are.

If there is sufficient time, look at one of the numbers whose placement and corresponding "yes" answer does not determine the strings in a single move and discuss the situation. For example, knowing where the number 4 is in the picture determines the green string but not the red string.

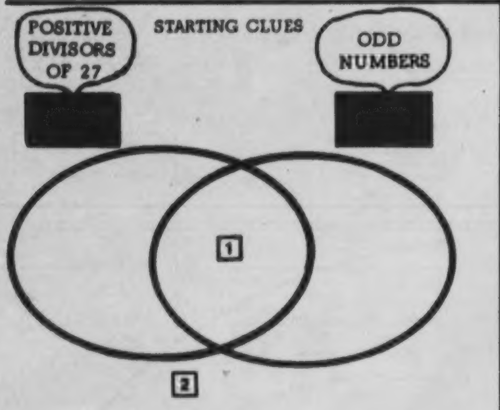
T: We still don't know what the two strings are so I'll now give you the final clue.

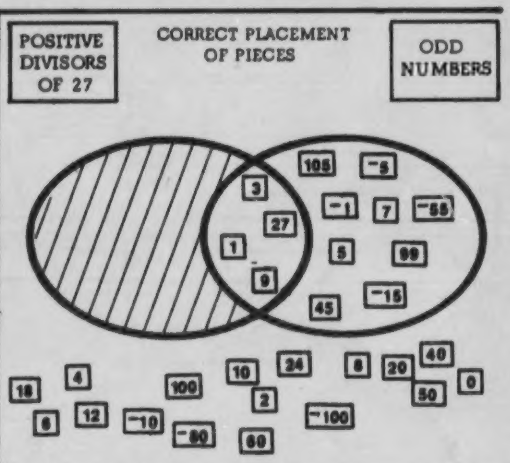
Put $\boxed{6}$ outside the three strings.

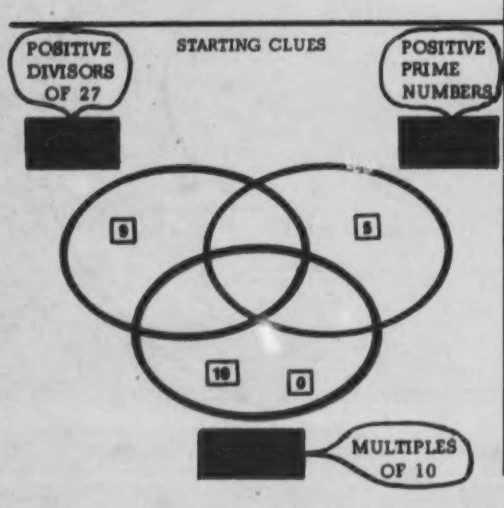
S: The red string is for the positive divisors of 27 and the green string is for the multiples of 10.

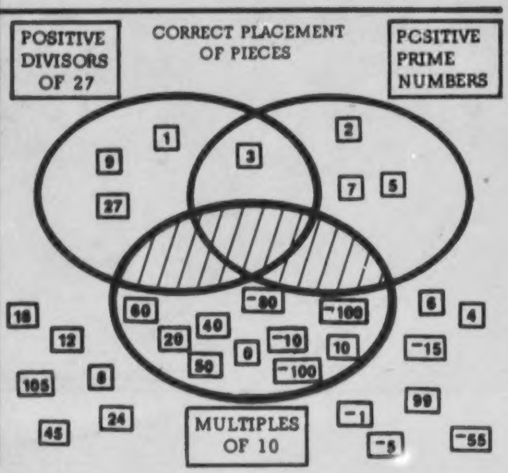
Exercise 2B

Play Version D of the String Game with numbers in the usual way. Two games with one or more empty regions are suggested below. Start with the first game and continue with the second game only if there is time and interest.


STARTING CLUES
POSITIVE DIVISORS OF 27
ODD NUMBERS
1
2


CORRECT PLACEMENT OF PIECES
POSITIVE DIVISORS OF 27
ODD NUMBERS
105
-5
3
27
1
9
-1
7
-55
5
99
45
-15
18
4
100
10
24
8
20
40
0
50
2
6
12
-10
-80
-100
60


STARTING CLUES
POSITIVE DIVISORS OF 27
POSITIVE PRIME NUMBERS
9
5
18
0
MULTIPLES OF 10


CORRECT PLACEMENT OF PIECES
POSITIVE DIVISORS OF 27
POSITIVE PRIME NUMBERS
1
9
27
3
2
7
5
60
-80
-100
40
20
0
-10
10
50
-100
18
12
105
8
45
24
6
4
-15
99
-1
-5
-55
MULTIPLES OF 10

L 10

1.4 Counting Problems #2

CAPSULE LESSON SUMMARY

Given a string picture, determine whether statements about the picture are true, false, or you can't tell. Include statements involving "exactly", "at most", and "at least". Given a statement, make additions to a string picture so that the statement is true. Extend the experiences in Lesson 1.2 to consider a set and three intersecting subsets.

MATERIALS

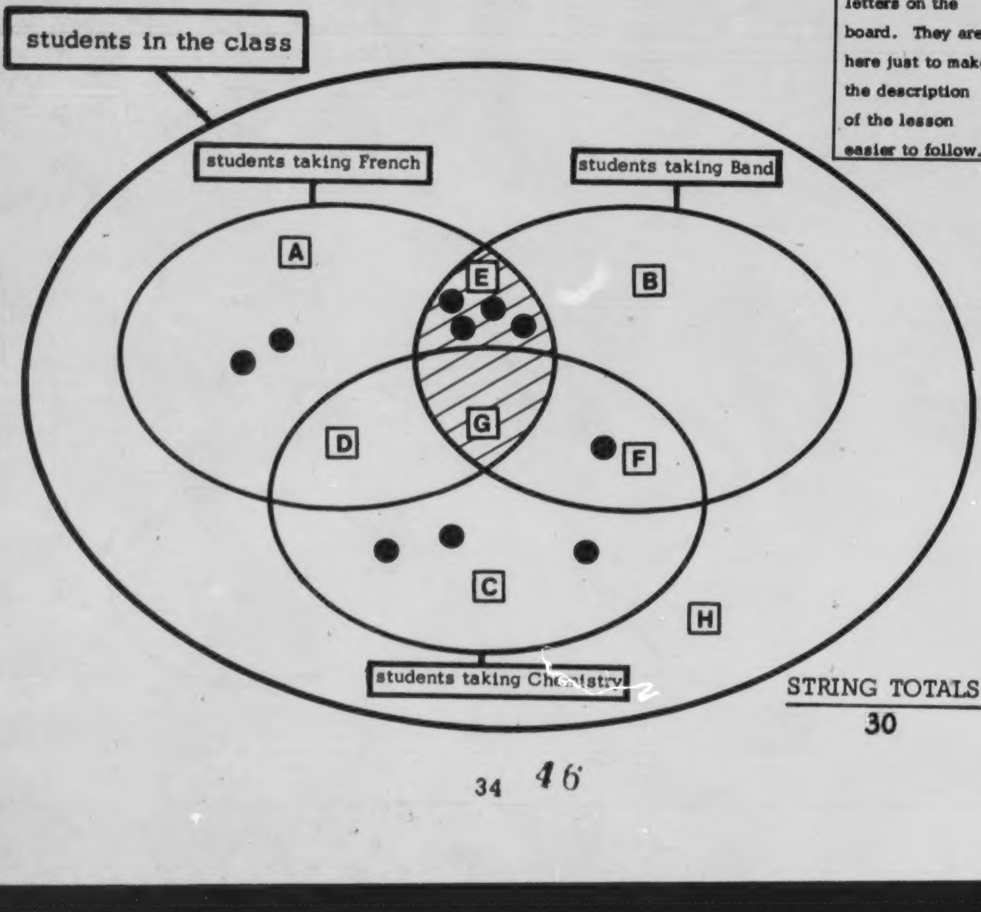
Teacher: Colored chalk
Student : Paper; colored pencils

DESCRIPTION OF LESSON

NOTE: If your class has not finished the previous Counting Problems lesson (Lesson 1.2), you should complete Exercises 3 and 4 of that lesson. In that case, you could do Exercise 1 and 2 of this lesson in any time that remains.

Exercise 1

Draw this string picture on the board.


students in the class
students taking French
students taking Band
A
E
B
G
D
F
C
H
students taking Chemistry

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

STRING TOTALS
30

T: I have a string picture on the board. It is about a class of thirty students in a high school. Some students take French, some take Band, and some take Chemistry. Is there a dot drawn in the picture for every student in the class?

S: No, there are less than thirty dots drawn.

T: That is right --- I have not shown you where all the dots should be. I am going to make some statements. Tell me if each statement is true, false, or you can't tell.

T: There are exactly six students in the class taking French.

S: Can't tell. There could be more than six.

T: Where . . . ?

Student points to regions A and D.

T: There are at least three Chemistry students in the class.

S: True. "At least three" means "three or more than three" and here (region C) are three dots and here (region F) is one more.

T: There are at most four Band students in the class.

S: False. "At most four" means "four or less than four" and here (region E) are four dots and here (region F) is one more and there could be more (in region B or F).

T: There are exactly four students in the class who take French and Band, but not Chemistry.

S: True. Here (region E) are four dots and the region is hatched.

T: There is at least one student in the class who takes French, Band, and Chemistry.

S: False. The middle (region G) is empty and hatched.

T:	At least two students in the class do not take Chemistry.	**S:**	True. These regions (A and E) have six dots in them.
T:	At least one student in the class takes none of these three subjects.	**S:**	Can't tell. This region (H) is not hatched nor does it have any dots in it. We also know that all the dots are not drawn.
T:	Every Band student in the class also takes French.	**S:**	False. There is a dot here (region F).
T:	There are no students in the class who take both French and Chemistry.	**S:**	Can't tell. This region (D) is not hatched nor does it have any dots in it.
T:	There are at most twenty students in the class who do not take any of these three subjects.	**S:**	True. There are ten dots drawn in the picture, none of which are here (region H). There are thirty students altogether and 30 - 20 = 10.

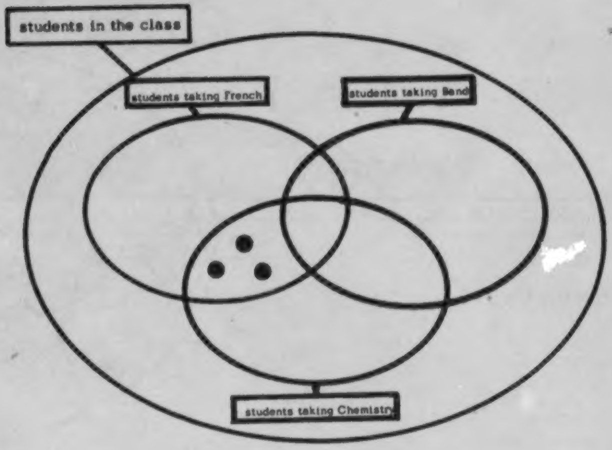
Exercise 2

Erase the dots and the hatching from the string picture of the previous exercise.

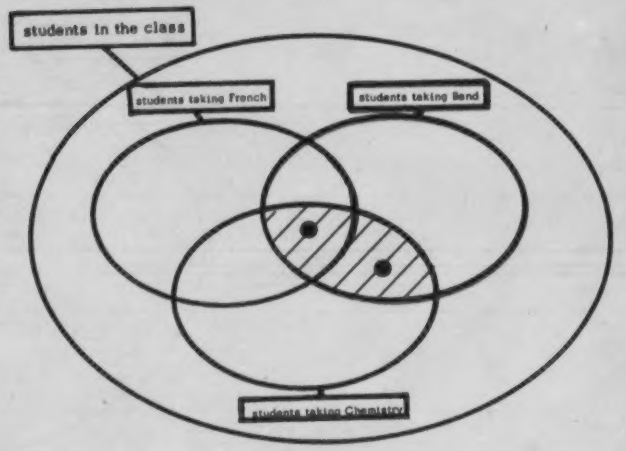
T: I will give you several statements about the string picture. Each time, I will ask someone to fix the picture to make the statement true. We will assume that there are thirty students in the class.

After each student response, let the class discuss whether or not it is correct and whether or not there are other possible ways to solve the problem. Before going on to a new statement, erase the dots and any hatching that was put in the picture. There are many correct responses for each statement; one possibility is shown at the right of each statement.

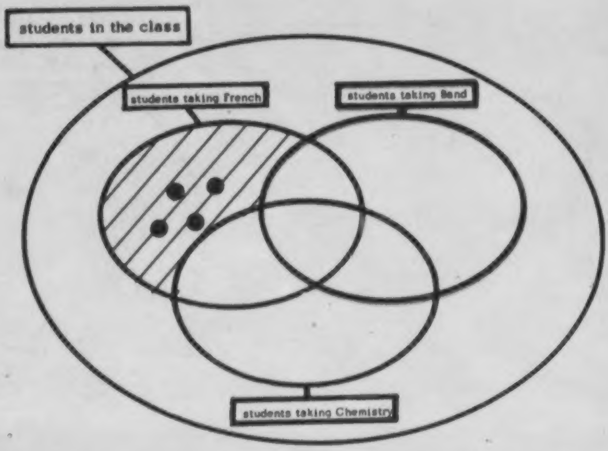
There are at least two Chemistry students who take French but not Band.


students in the class
students taking French
students taking Band
students taking Chemistry

There are at most three Band students in the class who take Chemistry.

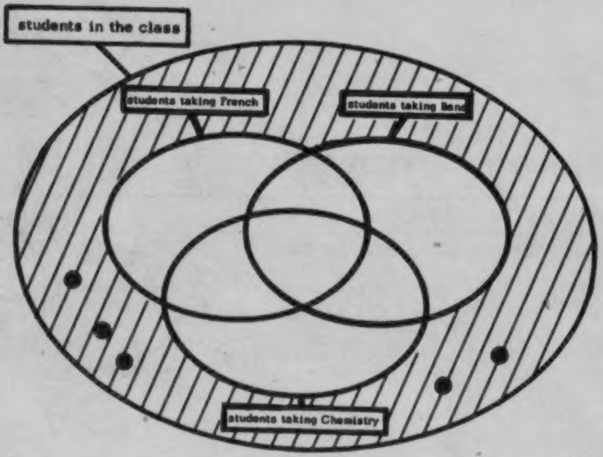

students in the class
students taking French
students taking Band
students taking Chemistry

There are exactly four students in the class who take French but neither Band nor Chemistry.

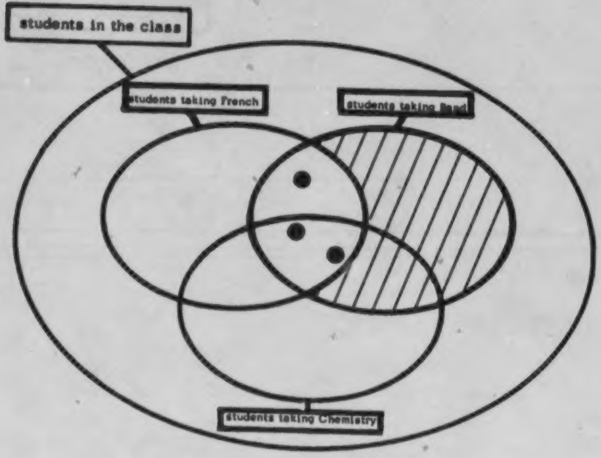

students in the class
students taking French
students taking Band
students taking Chemistry

L 10

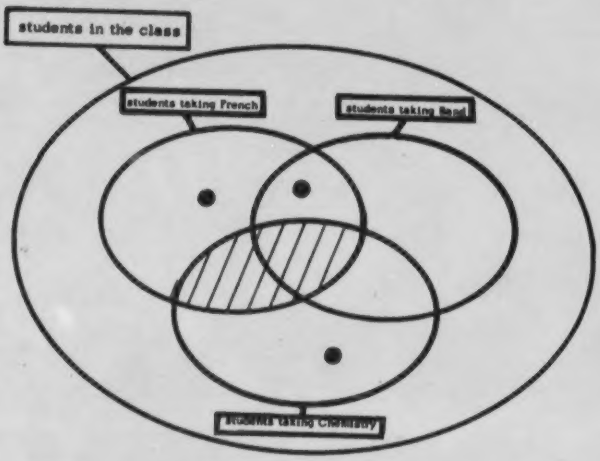
There are at most five students in the class who do not take any of these three subjects.


students in the class
students taking French
students taking Band
students taking Chemistry

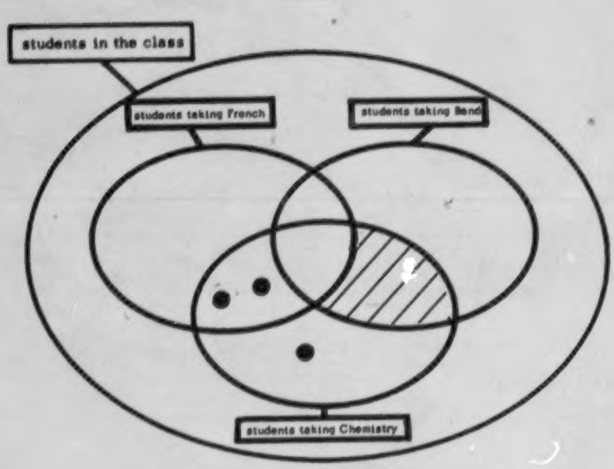
Every Band student in the class takes French.


students in the class
students taking French
students taking Band
students taking Chemistry

No student in the class who takes French also takes Chemistry.


students in the class
students taking French
students taking Band
students taking Chemistry

Optional:
Every Chemistry student in the class that takes Band also takes French.


students in the class
students taking French
students taking Band
students taking Chemistry

Exercise 3

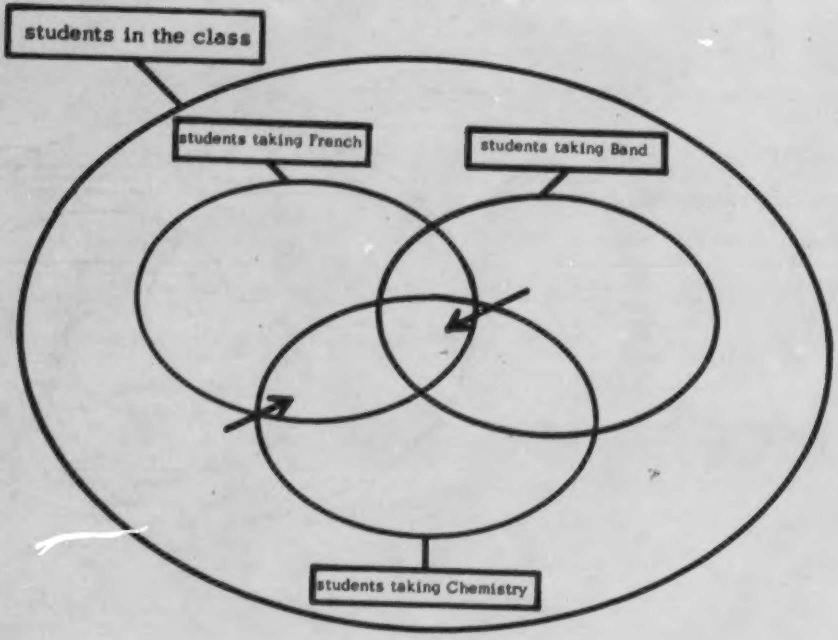
Erase the dots and any hatching from the string picture on the board. Distribute paper and colored pencils.

T: I am going to give you some clues to help you to figure out how many Band students there are in the class. After each statement tell me whether or not we can record in the picture the number of dots in any of the regions. Write down any information you wish to remember on your paper.

Allow time between statements to allow students to assimilate the information.

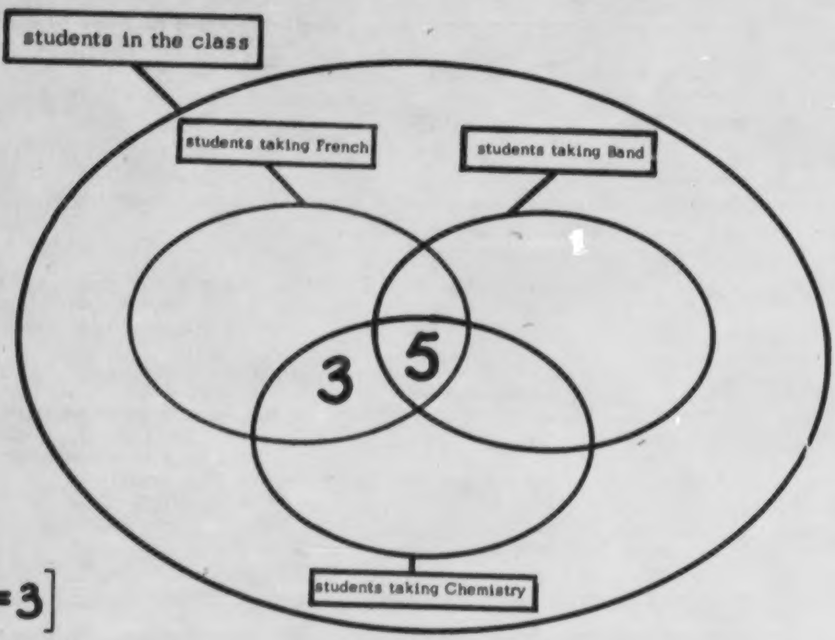
T:	S:
The number of students in the class who take French and Chemistry is eight.	No, we can not put any numbers in the picture. We know there are eight dots here (the intersection of the blue and red string), but we do not know how many are in each part (see arrows in the next illustration).

L 10


students in the class
students taking French
students taking Band
students taking Chemistry

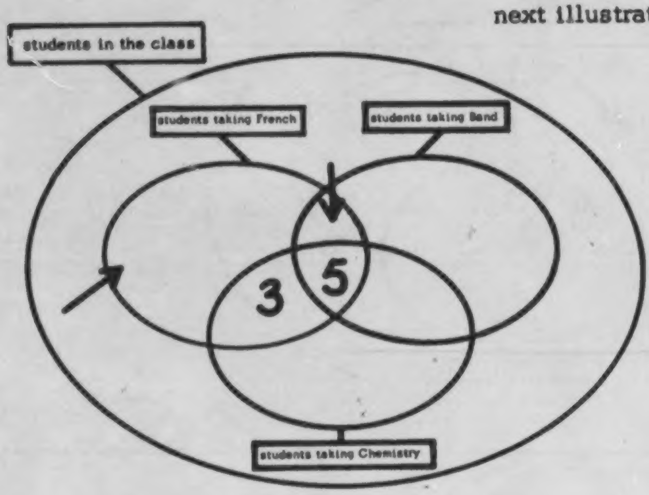
T: Five students in the class take all three of these subjects.

S: We can put two numbers in the picture.

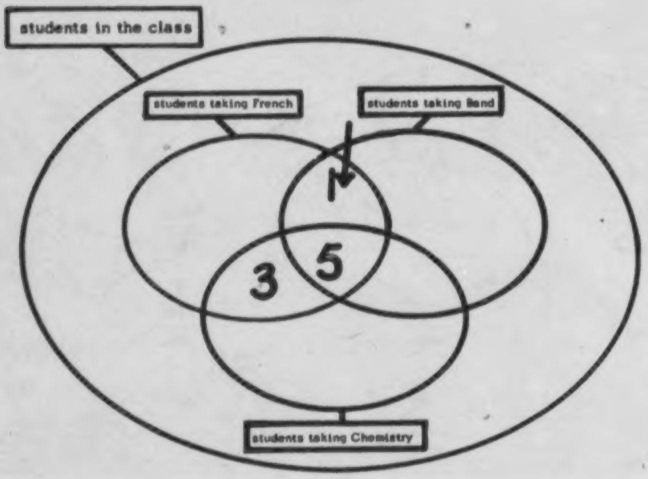

students in the class
students taking French
students taking Band
3
5
students taking Chemistry

[8−5=3]

T:	The number of students in the class who take French is fifteen.	**S:**	No, we cannot put any new numbers in the picture. We know where eight of the fifteen dots are, but the other seven could be in either of two places (see arrows in the next illustration).


students in the class
students taking French
students taking Band
students taking Chemistry
3
5

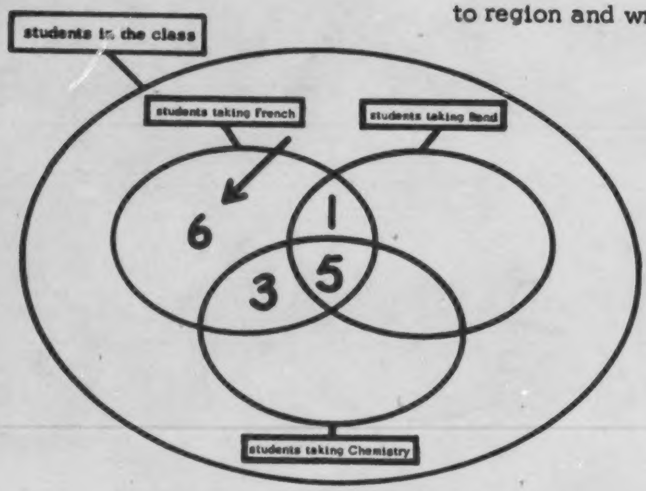
T:	The number of students in the class who take French and Band is six.	**S:**	We already know where five of the six students are so the other one must be here (see arrow in the next illustration).


students in the class
students taking French
students taking Band
students taking Chemistry
1
3
5

L 10

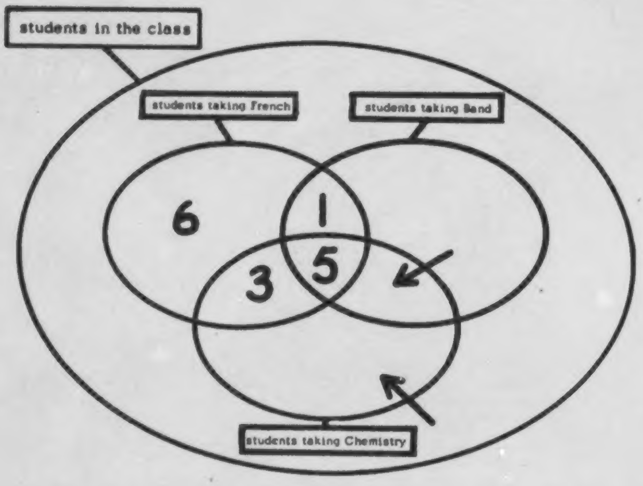
T: Can we put any other numbers in the picture?

S: We know there are fifteen students taking French. 15 - (1 + 3 + 5) = 6. There are six students here (points to region and writes "6").

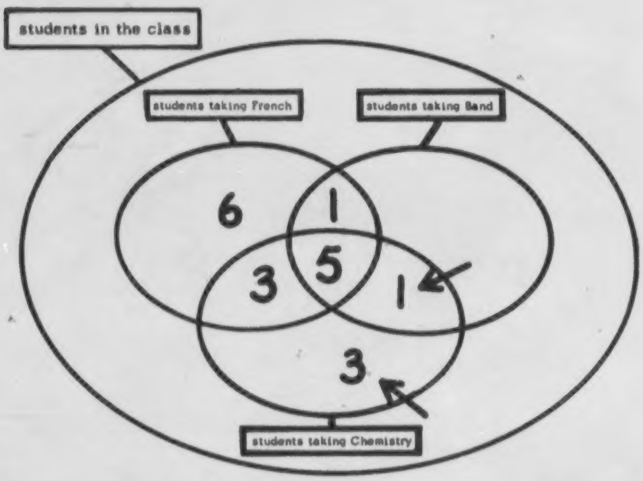

students in the class
students taking French
students taking Band
6
1
3
5
students taking Chemistry

T: Twelve students in the class take Chemistry.

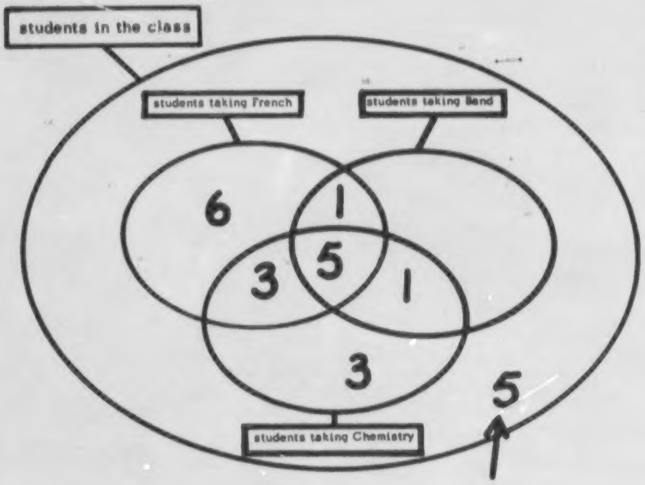
S: No, we cannot put any new numbers in the picture. We already know where eight of the twelve dots are in the red string, but the other four could be in either of two places (see the arrows in the next illustration).


students in the class
students taking French
students taking Band
6
1
3
5
students taking Chemistry

T:	There are three students in the class who take Chemistry but neither French nor Band.	S:	Three students are here (see illustration), and since there are twelve dots in the red string and 12 - (3 + 5 + 3) = 1, we know one dot is here (see illustration).


students in the class
students taking French
students taking Band
students taking Chemistry
6
1
3
5
1
3

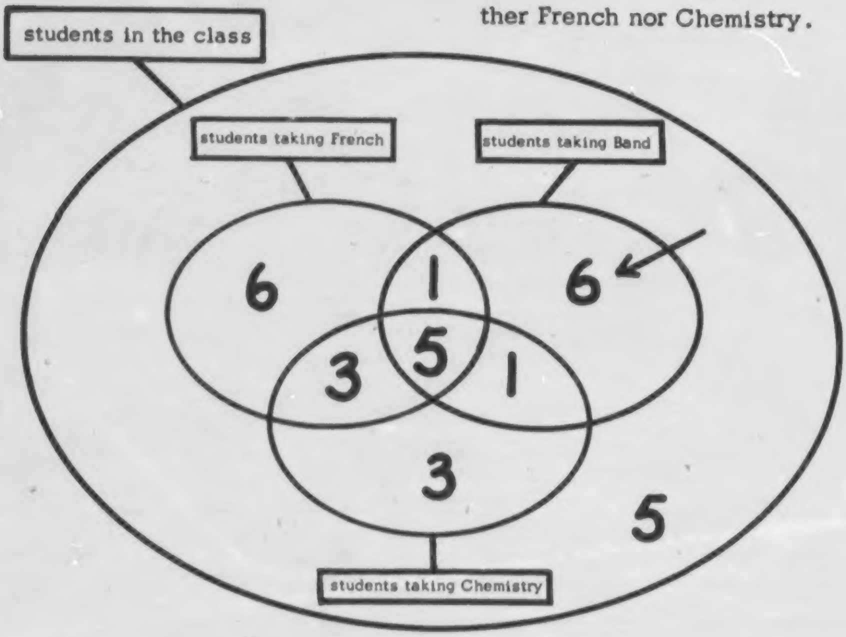
T:	There are five students who do not take any of these three subjects.	S:	These five students are here (see arrow in the illustration).


students in the class
students taking French
students taking Band
students taking Chemistry
6
1
3
5
1
3
5

L 10

T: There is another number we can put in the picture. Does anyone know this number and where to put it in the picture?

S: There are thirty students in the class and we have twenty-four (6 + 1 + 3 + 5 + 1 + 3 + 5 = 24) of them indicated in the picture. The other six must take Band but neither French nor Chemistry.


students in the class
students taking French
students taking Band
6
1
6
3
5
1
3
5
students taking Chemistry

T: How many students in the class take Band?

S: Thirteen, because 6 + 1 + 5 + 1 = 11.

1.5 String Game Analysis #3

CAPSULE LESSON SUMMARY

Play the String Game with three strings or simply place the game pieces in the picture with the string labels face-up. Present a three string picture where one string is for the positive prime numbers and the other two strings are each for the multiples of an unknown number. Discover the unknown numbers by the placement of various numbers in the string picture.

MATERIALS

Teacher:	Demonstration analysis sheets (two or three strings); colored chalk; Numerical String Game kit
Student :	String Game Analysis Pad (two or three strings); Worksheets L01, L02

DESCRIPTION OF LESSON

If your class would have difficulty playing the String Game with three strings, do Exercise 1B which involves three strings except the string cards are face up. (For many students, this kind of game will be enough of a challenge.)

Exercise 1A

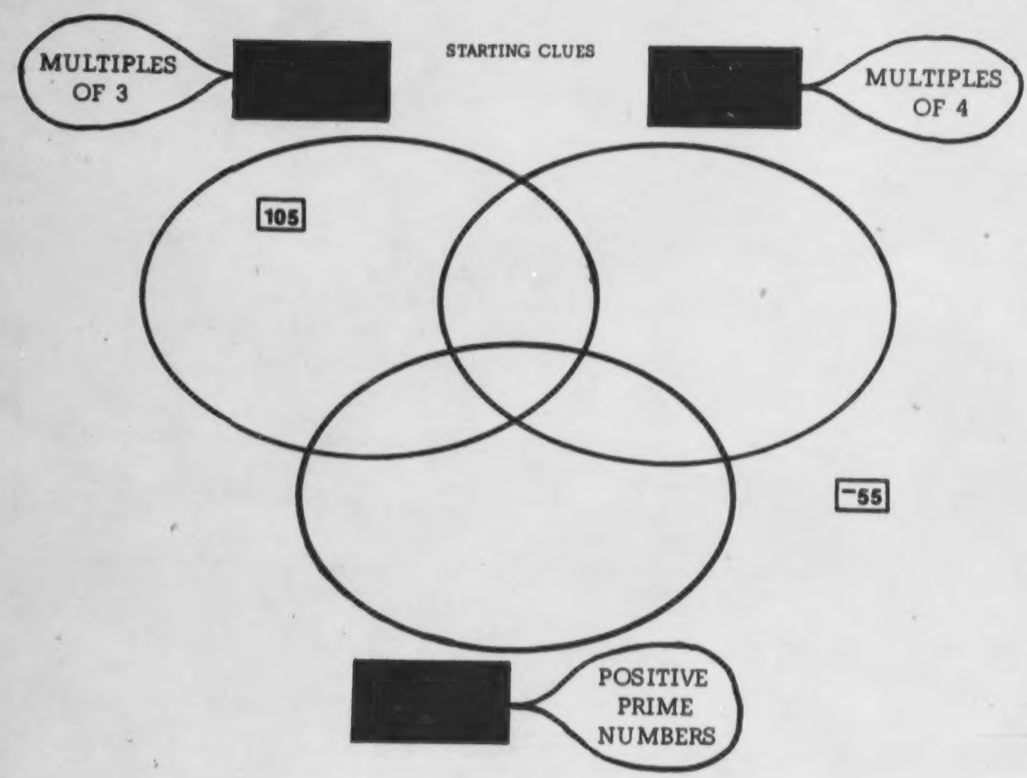
Play Version D of the String Game with Numbers in the usual way[*]. One of the strings should be for the positive prime numbers and the other two should each be for the multiples of some number. One possibility with two starting clues is suggested in the next illustration.

---

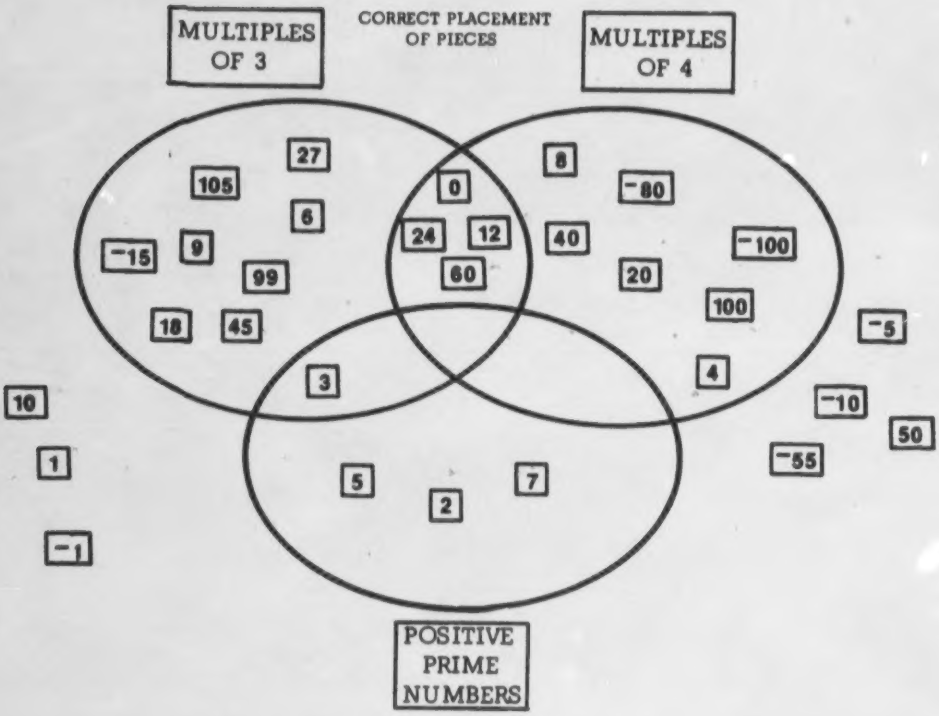
[*] See APPENDIX: THE STRING GAME for details on how to play different versions of the game.

L 13

STARTING CLUES


MULTIPLES OF 3
MULTIPLES OF 4
105
−55
POSITIVE PRIME NUMBERS

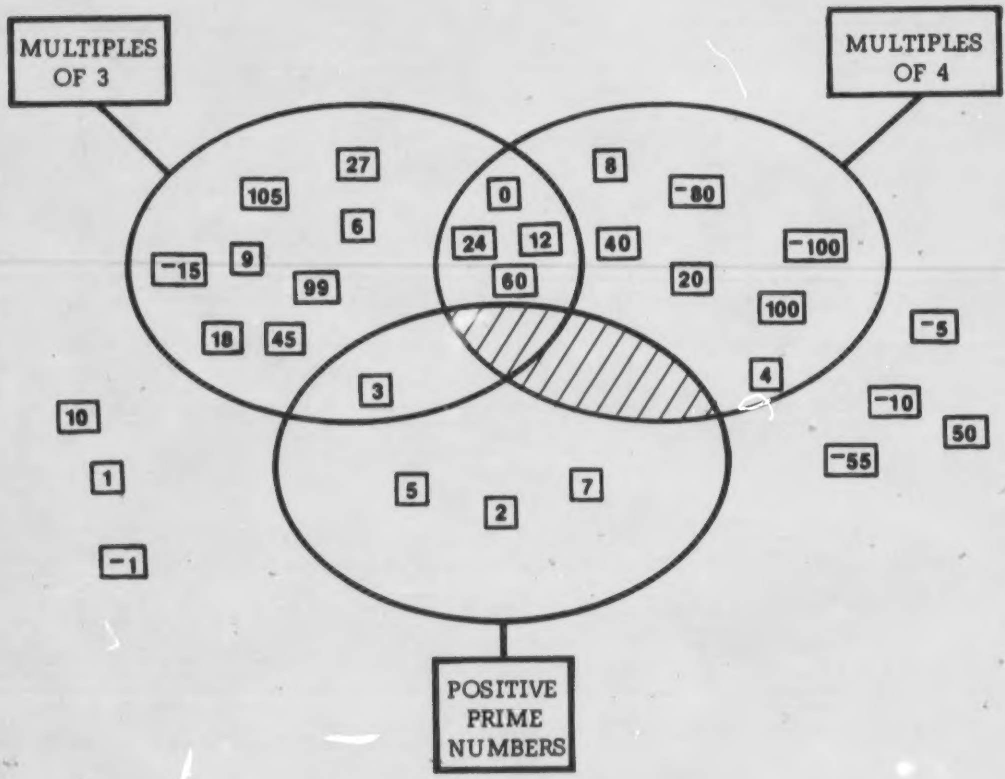
CORRECT PLACEMENT OF PIECES


MULTIPLES OF 3
MULTIPLES OF 4
27
105
6
−15
9
99
18
45
0
24
12
60
8
−80
40
20
−100
100
4
3
5
2
7
10
1
−1
−5
−10
50
−55
POSITIVE PRIME NUMBERS

When the game is finished, ask if there are any regions which should be hatched.

S (tracing the intersection of the blue and the green strings): There are no numbers here because no multiple of 4 is a prime number.

Hatch the two regions as shown in the next illustration.


MULTIPLES OF 3
MULTIPLES OF 4
POSITIVE PRIME NUMBERS
27
105
6
⁻15
9
99
18
45
0
24
12
60
8
⁻80
40
⁻100
20
100
4
3
5
2
7
10
1
⁻1
⁻5
⁻10
50
⁻55

L 13

Exercise 1B

Prepare your board as in Exercise 1A except turn the string cards face up. Teams take turns placing pieces just as in the String Game. The first team to place all of its pieces wins. Be sure that each team's set of playing pieces contains pieces from each of the strings. A division of pieces is suggested below.

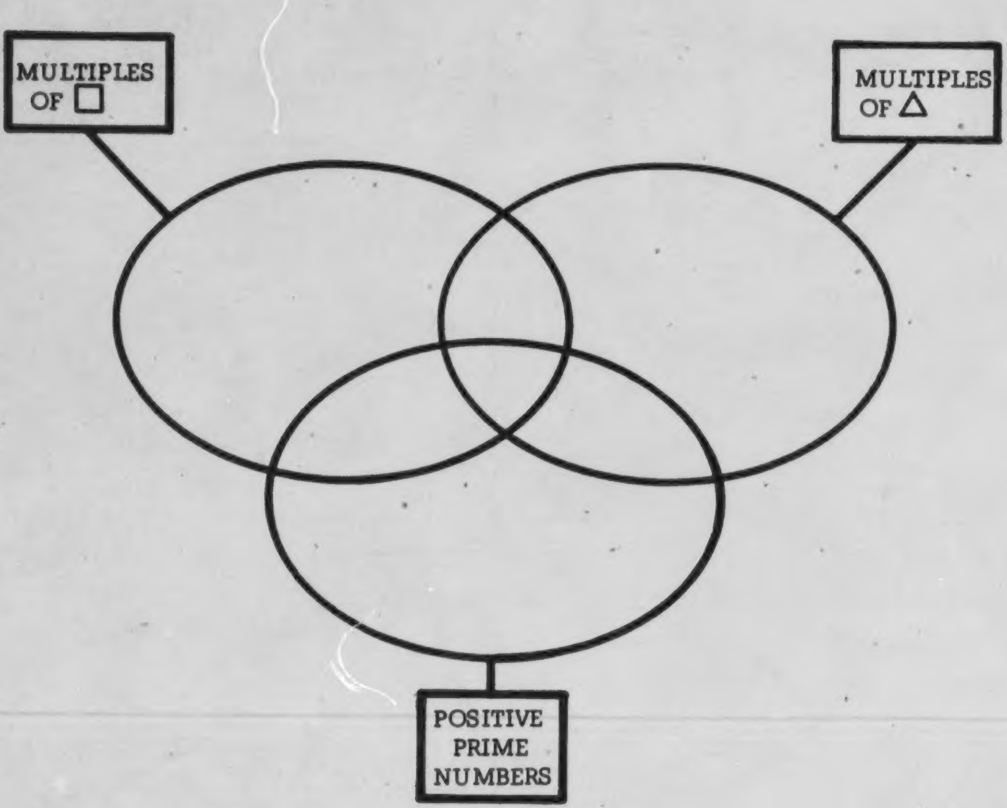
TEAM A			TEAM B		
-55	0	45	12	-15	1
24	10	3	60	4	50
105	-1	20	9	-10	27
6	99	8	2	-5	-80
-100	18	5	40	100	7

Exercise 2

Put away the String Game string cards and game pieces being very obvious so that the class realizes you are no longer only interested in string labels that occur in the game. Leave only the three strings on the board.

T: Suppose the green string is for positive prime numbers and the red string is for multiples of some number -- I put a "□" (box) for this number. And suppose the blue string is for multiples of some number -- I put a "△" (triangle) for this number.

Draw the string labels as in the next illustration.


MULTIPLES OF □
MULTIPLES OF △
POSITIVE PRIME NUMBERS

T: What numbers did we have for □ and △ in the game we just played?

S: 3 and 4.

T: I've chosen some new numbers for the box and the triangle, but I won't tell you what they are.

NOTE: Choose 5 for □ and 7 for △, but do not tell the class.

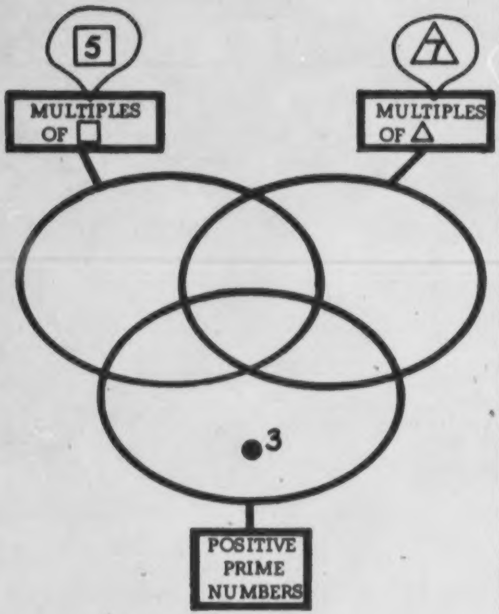
T: I want you to give me some numbers and I'll locate them in the picture. See how quickly you can guess what the strings are for.

L 13

We cannot predict what will happen in this exercise with your class, but we will include some dialogue from the lessons with our development classes.

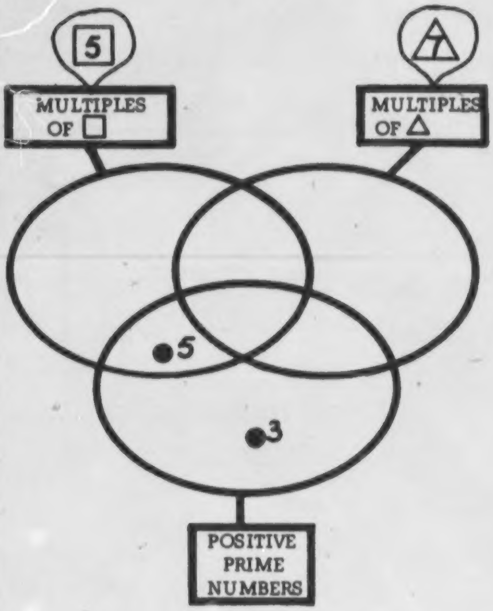
S: 3.

T: 3 goes here.


5
7
MULTIPLES OF □
MULTIPLES OF △
3
POSITIVE PRIME NUMBERS

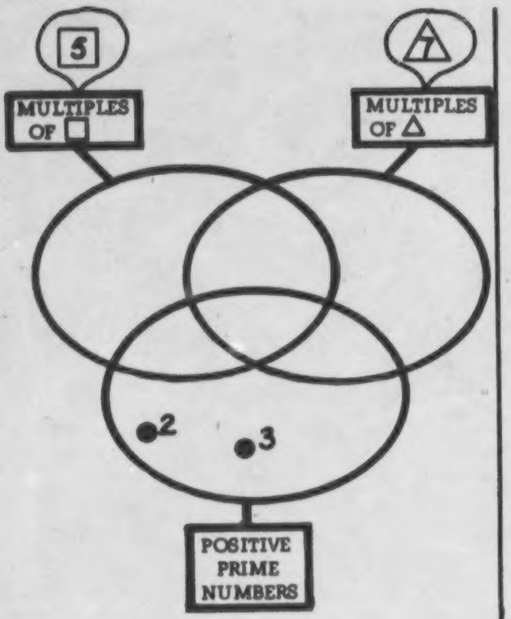
S: 5.

T: 5 goes here.


5
7
MULTIPLES OF □
MULTIPLES OF △
5
3
POSITIVE PRIME NUMBERS

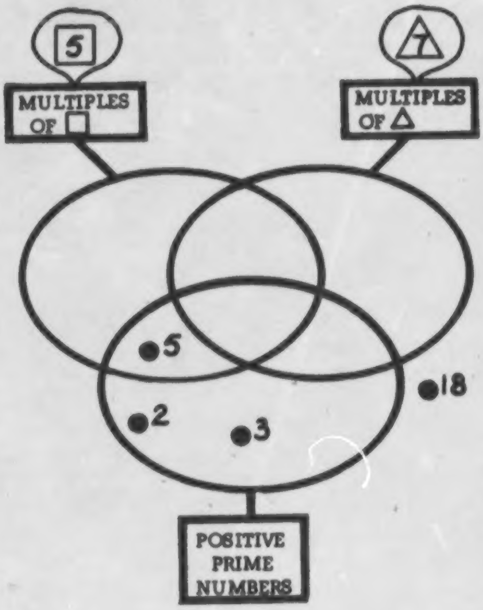
S: 2.

T: 2 goes here.


5
7
MULTIPLES OF ☐
MULTIPLES OF △
2
3
POSITIVE PRIME NUMBERS

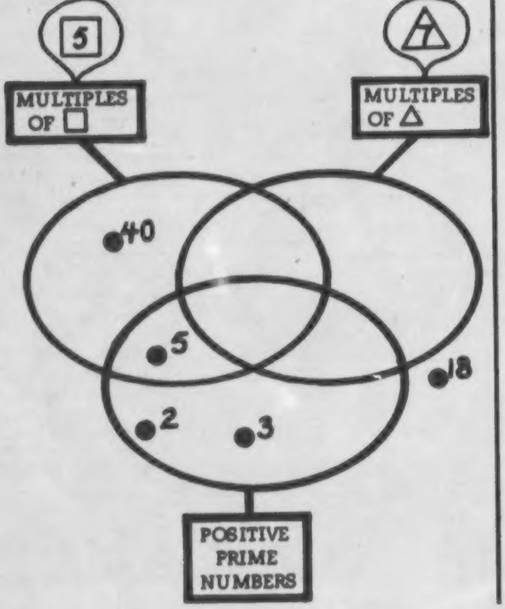
S: 18.

T: 18 goes here.


5
7
MULTIPLES OF ☐
MULTIPLES OF △
5
18
2
3
POSITIVE PRIME NUMBERS

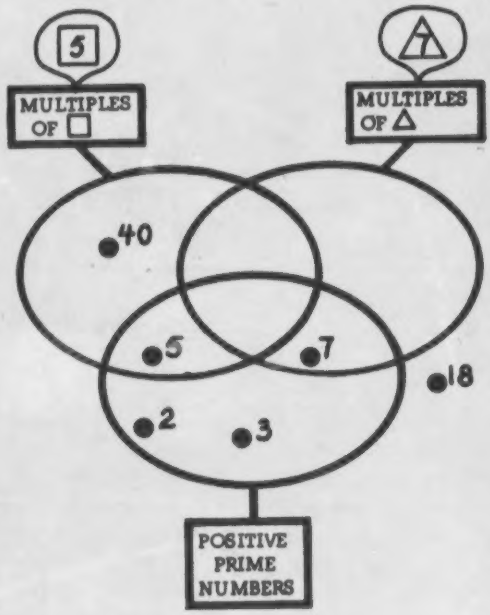
S: 40.

T: 40 goes here.


5
7
MULTIPLES OF ☐
MULTIPLES OF △
40
5
18
2
3
POSITIVE PRIME NUMBERS

S: 7.

T: 7 goes here.

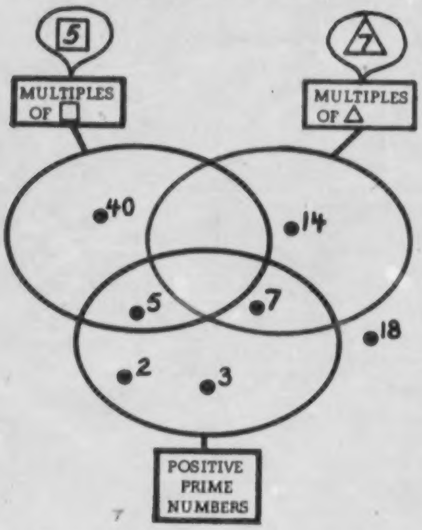

5
7
MULTIPLES OF ☐
MULTIPLES OF △
40
5
7
18
2
3
POSITIVE PRIME NUMBERS

L 13

At this point the strings are "determined" and several students indicate that they know what the strings are for.

T: If you know what the strings are for, where does 14 belong?

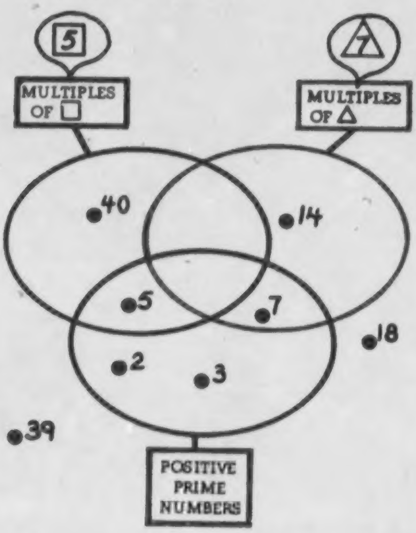
A student comes to the board and puts 14 in the picture.


5
7
MULTIPLES OF □
MULTIPLES OF △
40
14
5
7
18
2
3
POSITIVE PRIME NUMBERS

T: That is correct. Someone give us another number and I'll call on another person to put it in the picture.

S: 39.

A student correctly places 39 in the picture.

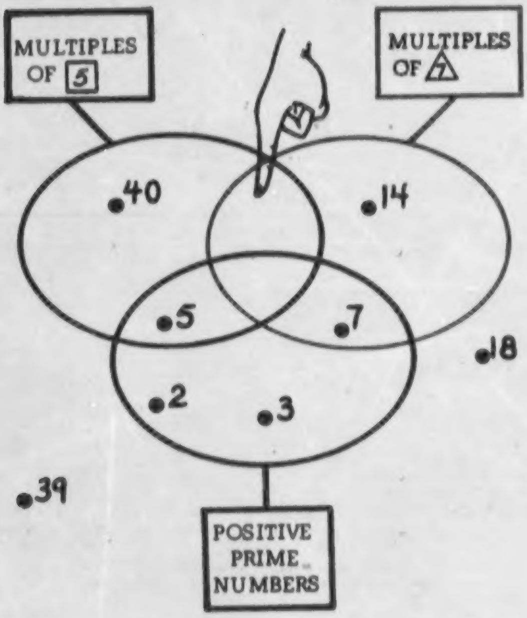

5
7
MULTIPLES OF □
MULTIPLES OF △
40
14
5
7
18
2
3
39
POSITIVE PRIME NUMBERS

T: What are the strings for?

S: The red string is for multiples of 5 and the blue string is for multiples of 7.

The teacher then writes "5" in the box and "7" in the triangle.

T: Are there any numbers which belong here? (See the next illustration.)

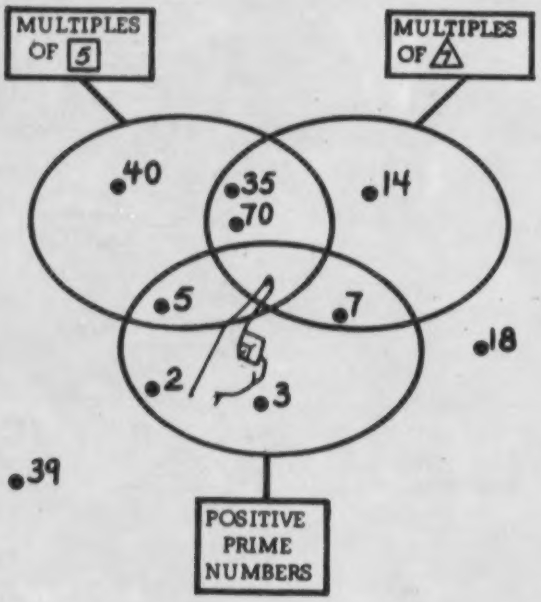

MULTIPLES OF 5
MULTIPLES OF 7
40
14
5
7
18
2
3
39
POSITIVE PRIME NUMBERS

S: 70.

S: 35.

L 13

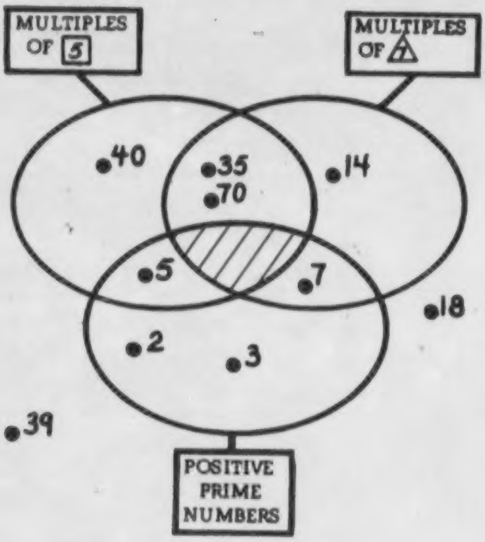
T: Are there any numbers which belong here? (See the next illustration.)


MULTIPLES OF 5
MULTIPLES OF 7
40
35
70
14
5
7
18
2
3
39
POSITIVE PRIME NUMBERS

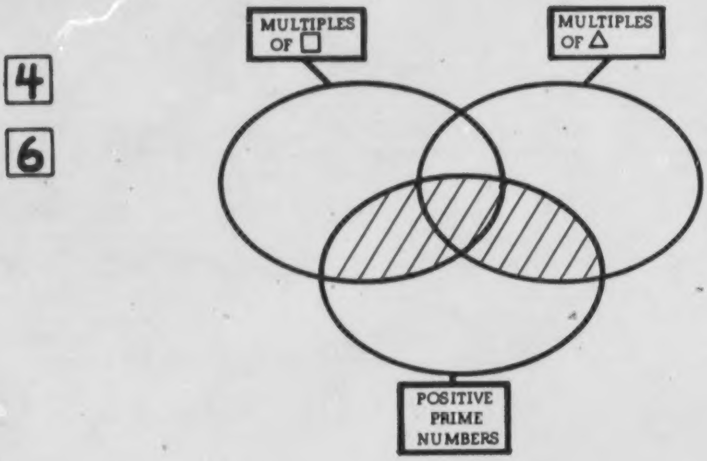
S: No, because a number which is both a multiple of 5 and a multiple of 7 cannot be a prime number.

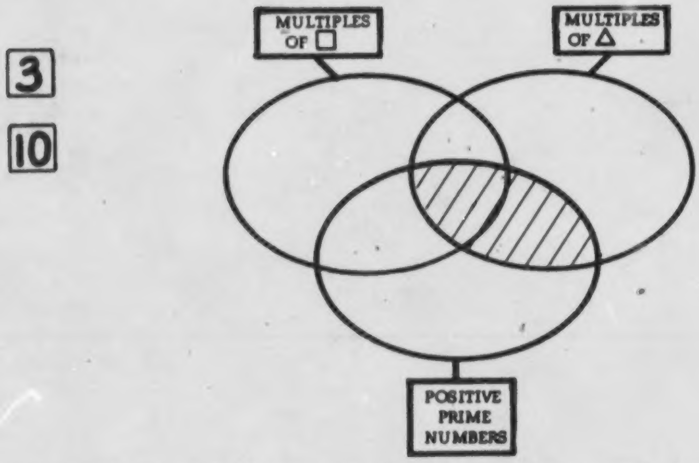
T: How can I indicate that on the picture?

S: Hatch it.


MULTIPLES OF 5
MULTIPLES OF 7
40
35
70
14
5
7
18
2
3
39
POSITIVE PRIME NUMBERS

Repeat this activity several times. Two games are suggested below. The diagrams show you which regions are empty and hatched. They should be hatched only after the strings have been identified.


4
6
MULTIPLES OF □
MULTIPLES OF △
POSITIVE PRIME NUMBERS

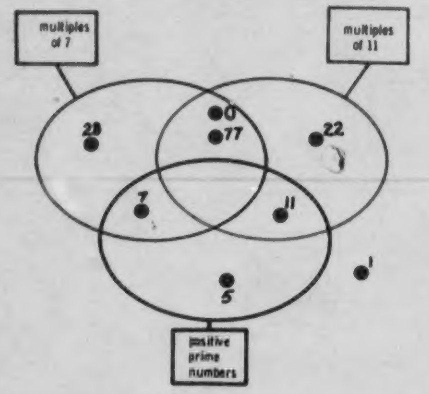

3
10
MULTIPLES OF □
MULTIPLES OF △
POSITIVE PRIME NUMBERS

Worksheets L01 and L02 are available for independent work for the remainder of the lesson. If some of your students have difficulty with Worksheet L02, suggest they start putting whole numbers in the picture in succession --- 0, 1, 2, 3, ...

Name ______________________ L 01 *

Label each dot in the string picture with one of these numbers.

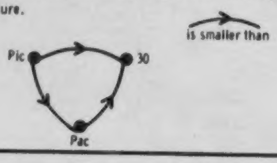
0 1 5 7 11 22 28 77

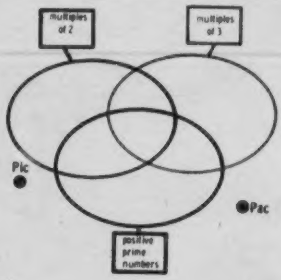

multiples of 7
multiples of 11
28
0
77
22
7
11
5
1
positive prime numbers

Which numbers, if any, belong in the middle region? None

Name ______________________ L 02 **

Pic and Pac are secret whole numbers. They are in this arrow picture and in this string picture.


is smaller than
Pic
30
Pac


multiples of 2
multiples of 3
Pic
Pac
positive prime numbers

Pic is 1.

Pac is 25.

2

IMAGES

The notion of image was first introduced in Part III of the CSMP Intermediate Grades Program by way of exploring some geometric relations, namely parallel projections and translations. Here the concept of the image of a set under a relation appears in a more general setting as well as in situations involving specific numerical and nonnumerical relations. The idea is simple: given a relation R and a set A, the image of A under the relation R, denoted "R(A)", is the set of ending points of R-arrows which have their starting points in A. A variety of relations: abstract, numerical, family, and geometric, provide the settings in which to examine the notion of the image of a set. In some cases, iterated images are of interest; that is, one determines R(A), then RR(A), then RRR(A), and so on. A reverse problem: given a relation and a set, determine some possibilities for a set with the given set as its R-image, presents an interesting study in the geometric situations.

CAPSULE LESSON SUMMARIES

Section 2.1:	Introduce the concept of the image of a set under a relation R. Consider several arrow pictures: one where the relation is not identified, one where the relation is "you are my sister", and (optional) one where the relation is "is smaller than". (Worksheet L03)
Section 2.2:	Extend the experiences in Section 2.1. Consider an arrow picture where the relation is not identified and one where the relation is +5 and the dots are labeled. Identify the numbers in the image of a finite set of numbers under a numerical relation. (Worksheets L04, L05, L06)

Section 2.3: Play a Guess My Rule game where the rule is for a function R on the set of points in the plane. Apply this rule to find the R-images of a variety of sets of points. Given a set, determine some possibilities for a set of points with the given set as its R-image.
(Worksheets L07, L08)

Section 2.4: Examine the composition of a 2x relation with itself one or more times. Find R(W), RR(W), RRR(W),...., where W is the set of whole numbers and where R is the relation 2x.
(Worksheets L09, L10, L11)

Section 2.5: Play a Guess My Rule game where the rule is for a function R on the set of points in the plane. Apply this rule to find the R-images of a variety of sets of points. Given a set, determine some possibilities for a set of points with the given set as its R-image.
(Worksheets L12, L13, L14, L15, L16)

2.1 Images #1

CAPSULE LESSON SUMMARY

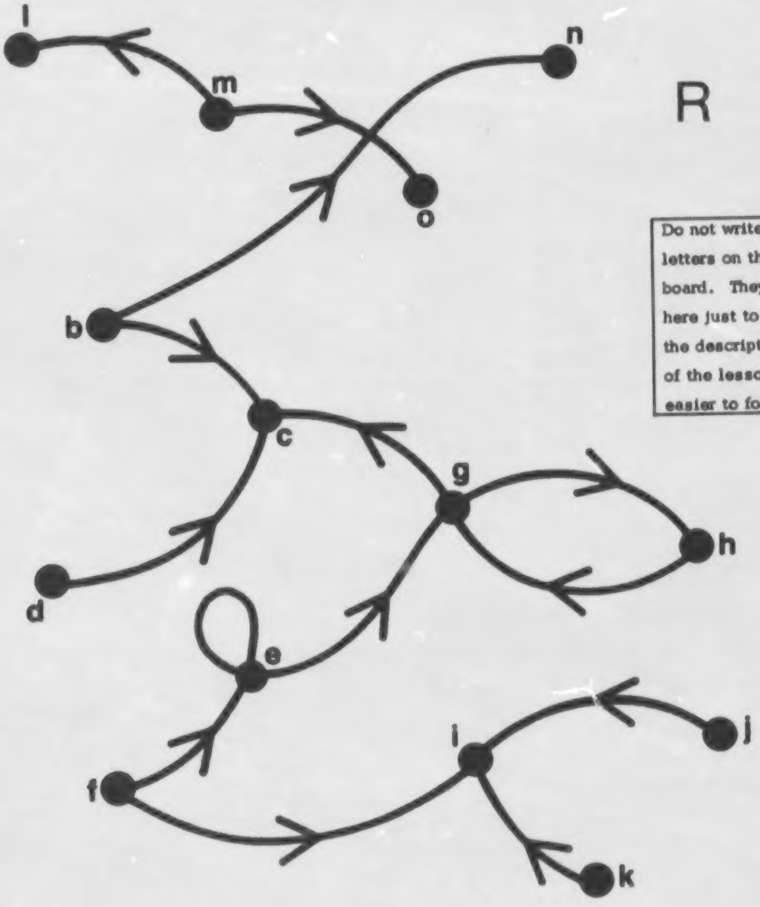
Introduce the concept of the image of a set under a relation R. Consider several arrow pictures : one where the relation is not identified, one where the relation is "you are my sister", and (optional) one where the relation is "is smaller than".

MATERIALS

Teacher : Colored chalk
Student : Worksheet L03 ; colored pencils

DESCRIPTION OF LESSON

Draw this relation very carefully on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

L 3

T: On the board I have drawn a relation with red arrows. Is it a function?

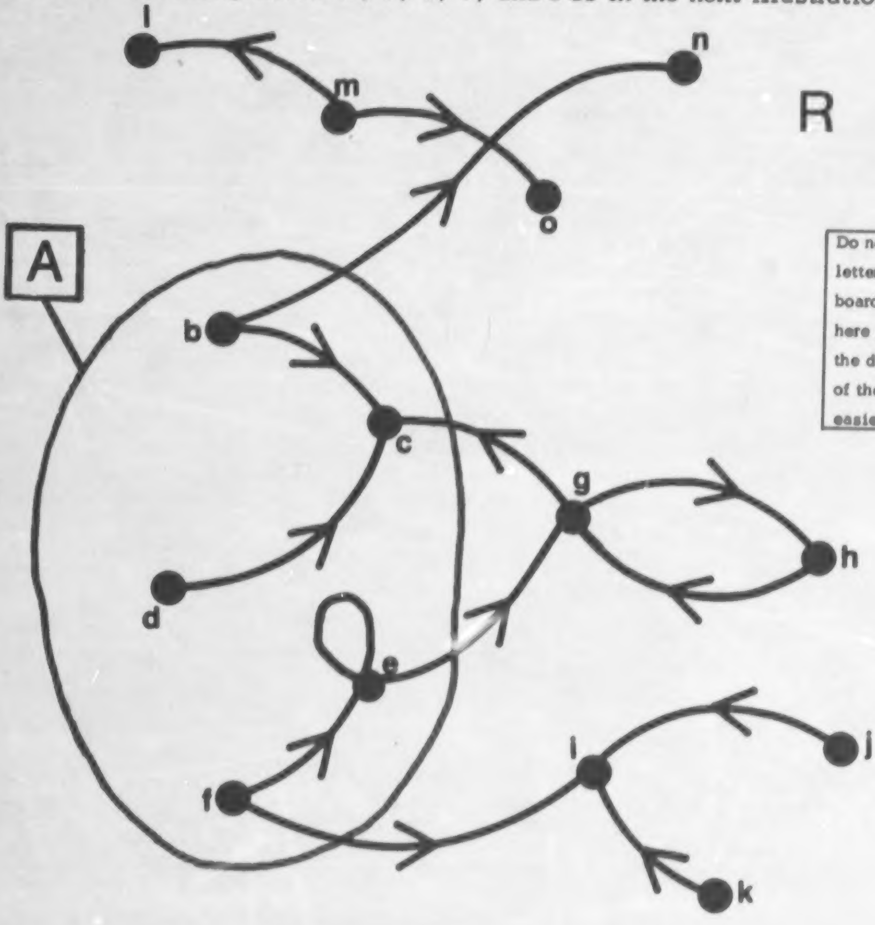
S: No, because a function has at most one arrow starting at each dot. At some of these dots, two arrows start.

You may need to help students with the definition of a function.

T: If it is not a function, can you show me a guilty dot?

Call on students, one at a time, to point to guilty dots. Each time ask the student to trace more than one arrow starting from the suggested guilty dot. There are five guilty dots in the picture: m, b, g, e, and f.

Draw a blue string around b, c, d, e, and f as in the next illustration.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: A is the set of dots in the blue string.

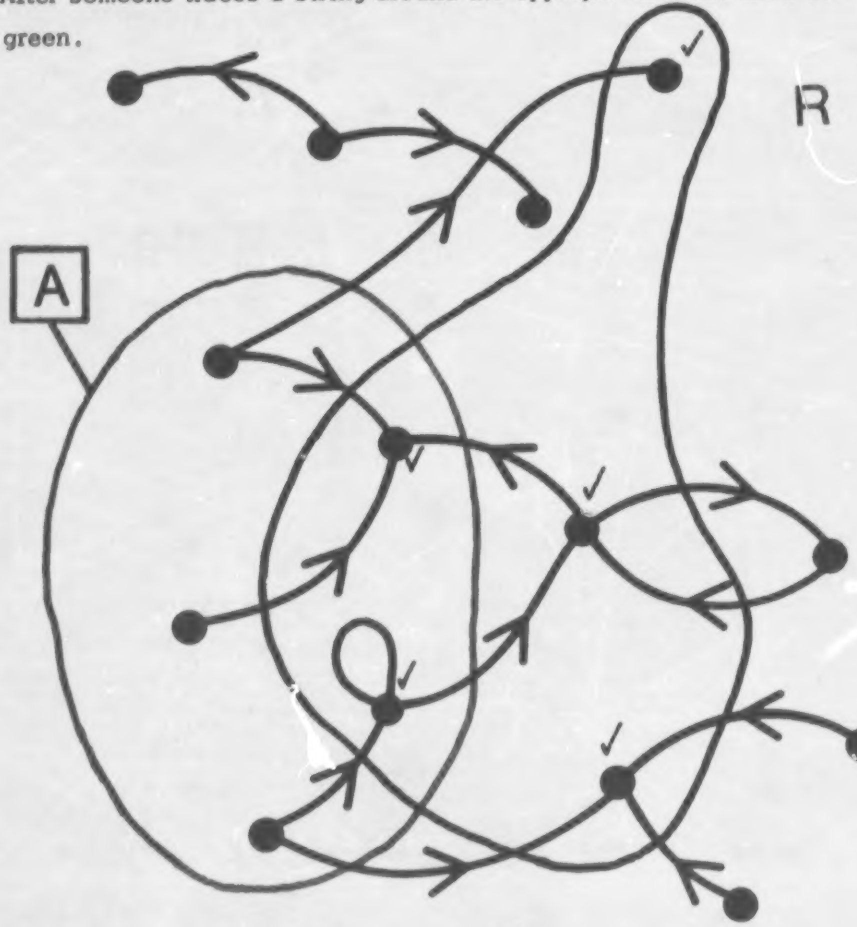
Trace the blue string.

T: If I put my finger on a dot in A and trace an arrow starting from that dot, where could I end?

Invite students one by one to trace arrows starting in A and then to point to their ending dots. Put a check (√) by each correct ending dot. When finished, you should have checks by n, c, e, g, and i.

T: We have checks by all of the dots which are ending dots of arrows starting in A. Can someone trace a string around _just_ these dots with checks?

After someone traces a string around the appropriate dots, draw the string in green.



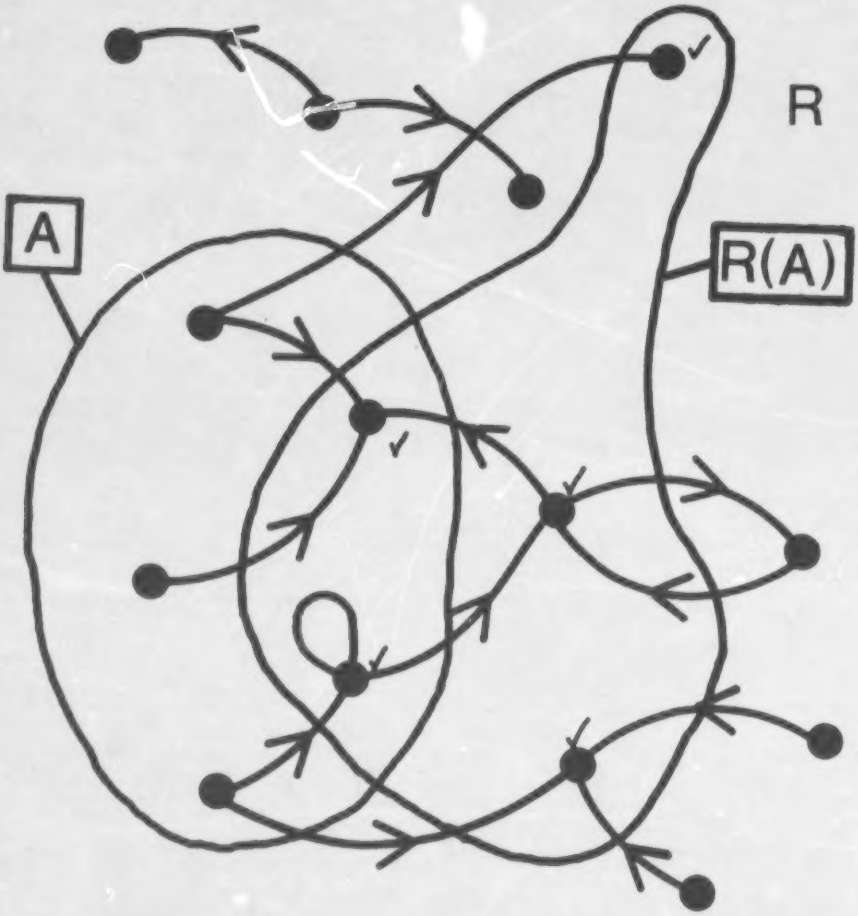
L 3

T: In the green string we have all the ending dots of arrows which have their starting dots in A. We call this set of dots the R-image of A.

Write this name on the board.

R-image of A

Label the green string "R(A)", indicating to the class that this is an abbreviation for "the R-image of A".



71

Write "R(A)" under "R-image of A" on the board.

R-image of A

R(A)

NOTE: If your class has average or better students, continue the discussion of this relation as is suggested in Exercise 2-A. If most of your students are weak, they would benefit from a similar activity with other relations as suggested in Exercise 2-B.

Exercise 2-A

Keep the picture from Exercise 1 on the board.

T: R(A) (read "R of A") is the set of dots in the green string. What do you think RR(A) (read "R of R of A") is?

On the board under "R(A)" write "RR(A)".

R-image of A

R(A)

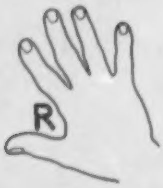
RR(A)

S: The image of the green string.

S: The image of the image of A.

L 3

Cover the "R (A)" in "RR (A)" written on the board.

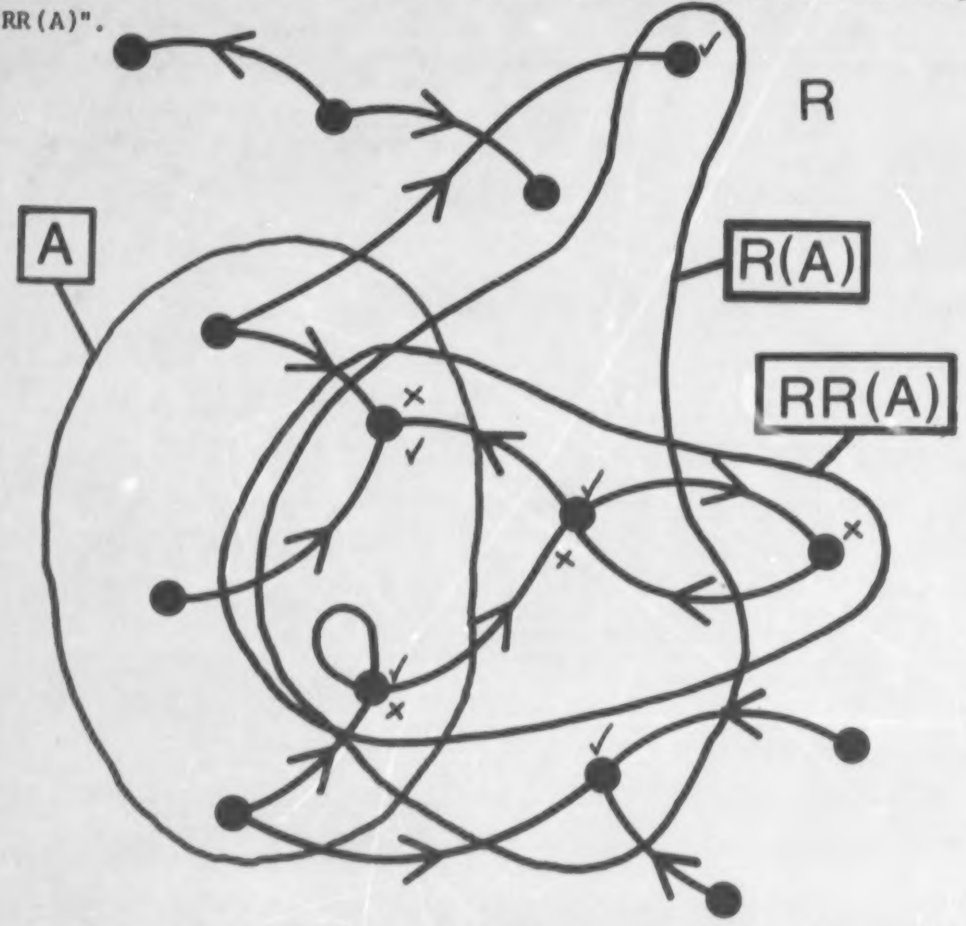


T: We want the image of (lift your hand) R (A) (trace the green string). Who can show us an ending dot of an arrow starting in the green string; that is, a dot in RR (A)?

•

Invite students one by one to trace an arrow starting in R (A) (i.e., in the green string) and then to point to its ending dot. Mark (x) each correct ending dot. You will have cross marks by h, c, e, and g.

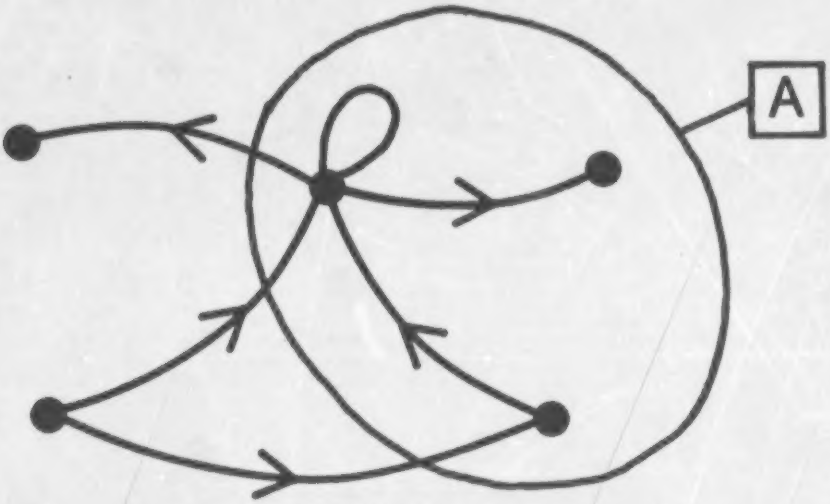
Draw a string around these dots as in the next illustration and label the string "RR (A)".



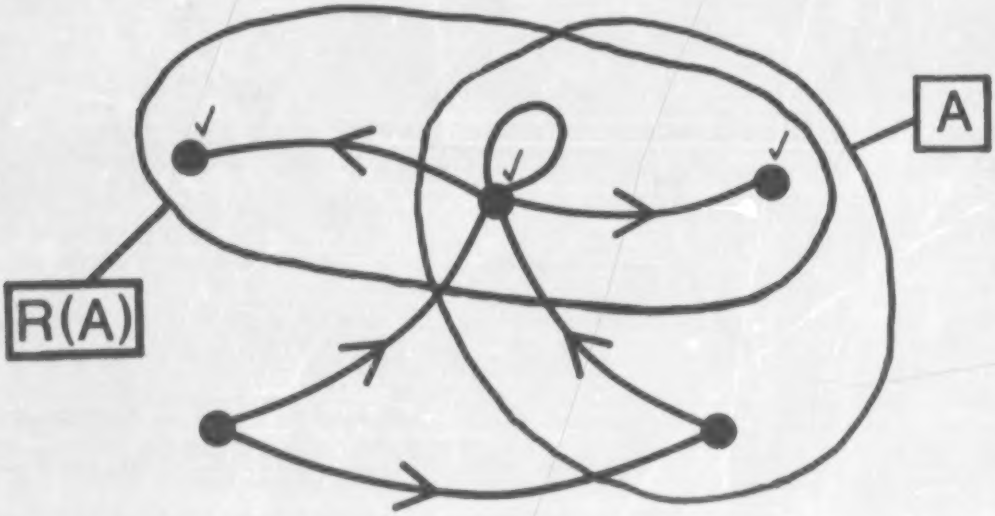
If your students are doing well with this activity, ask them what RRR(A) is (i.e., the R-image of RR(A)). It turns out that you get the dots in the black string again. Conclude: RR(A) = RRR(A).

Exercise 2 - B

Erase the picture from Exercise 1 and draw this one on the board.

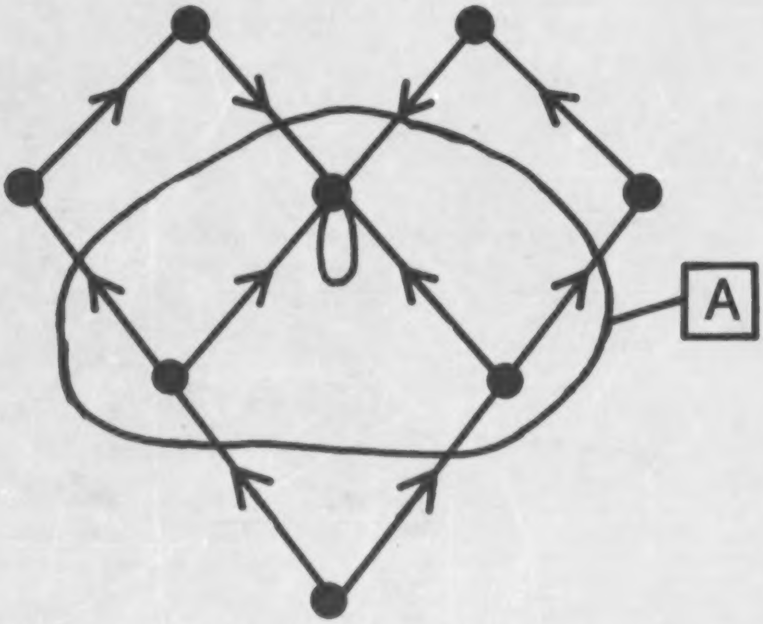


Ask the students to find R(A) as in Exercise 1.

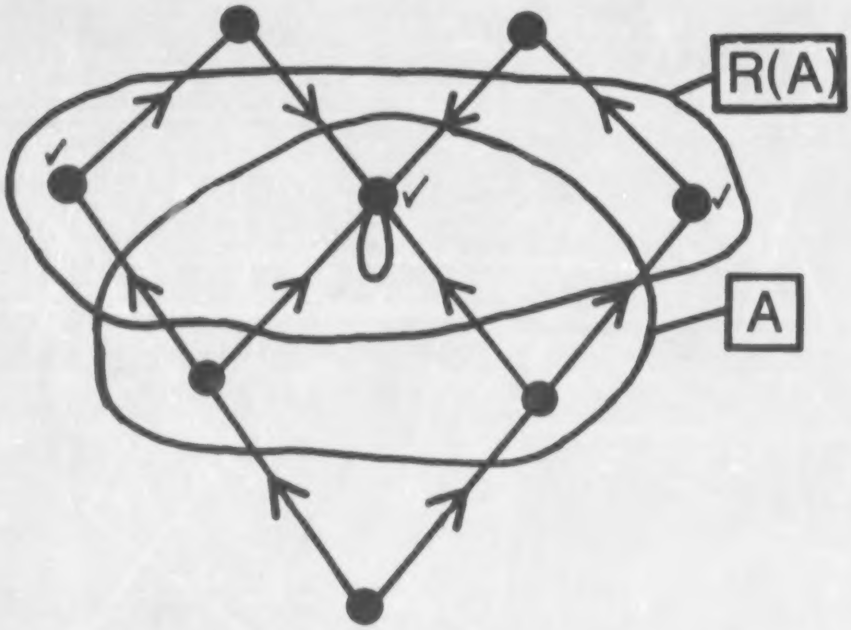


L 3

Erase the picture and draw this one on the board.

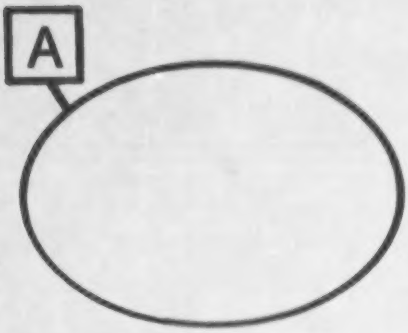


Ask the students to find R(A) as in Exercise 1.



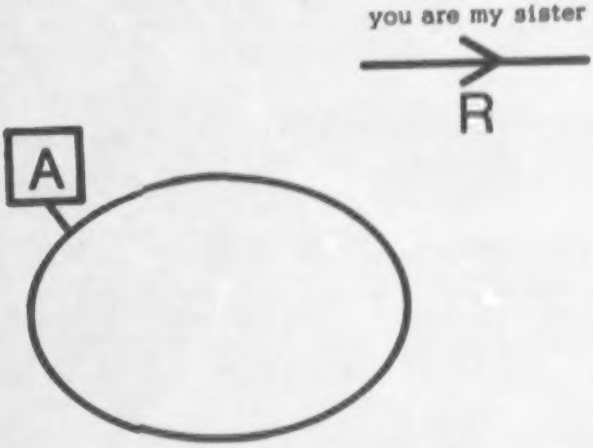
Exercise 3

Draw a large blue string on the board and label it "A".



T: Let's look at another relation.

Draw a red key arrow for "you are my sister". Write " R " below the arrow.

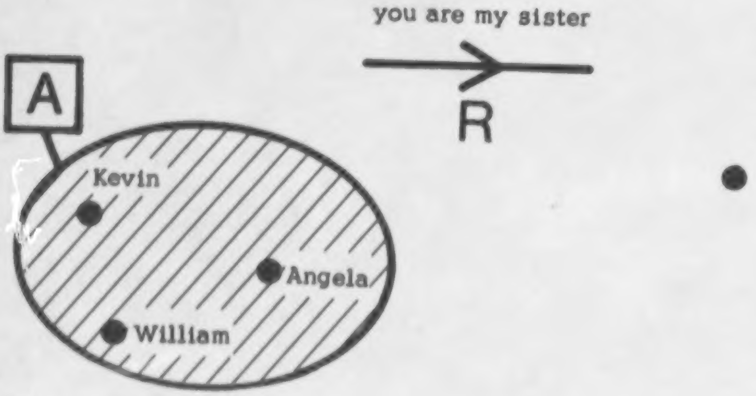


Choose three or four students from your class -- try to choose someone who has no sisters, someone who has one sister, and someone who has more than one sister. In the dialogue, we will assume their names are "Angela", "Kevin", and "William".

L 3

T: The red arrows are for "you are my sister".

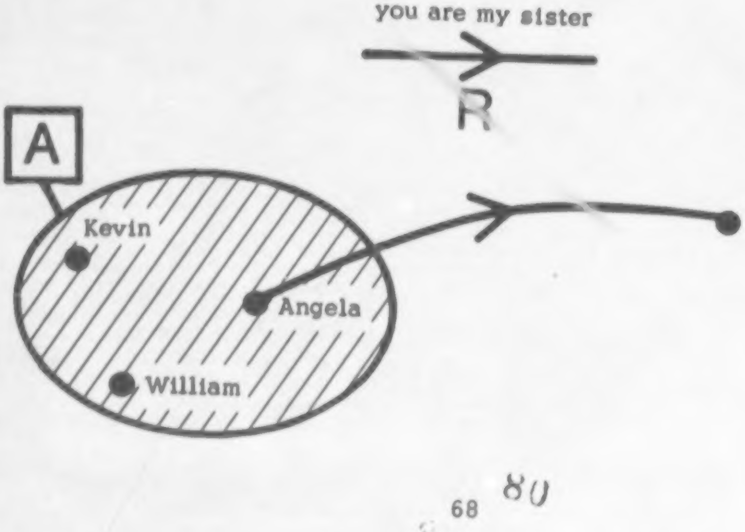
Draw a dot in the blue string for each of the students you have chosen and label them. Hatch the blue string.


you are my sister
R
A
Kevin
Angela
William

T: The only students who will be in the blue string are Kevin, Angela, and William. Angela, do you have a sister?

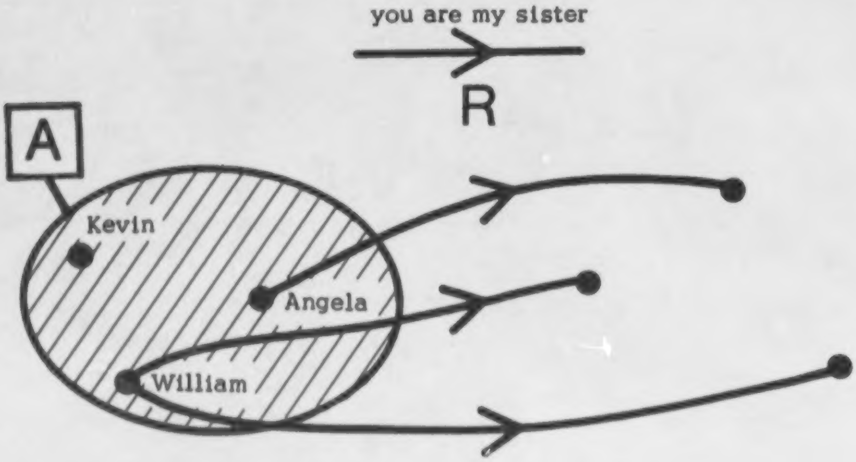
Angela: Yes, her name is "Kim".

Draw a dot outside the blue string. Draw a red arrow between the two dots as in the next illustration.


you are my sister
R
A
Kevin
Angela
William

Continue this activity asking the other students in the blue string if they have sisters and recording the information in the picture.

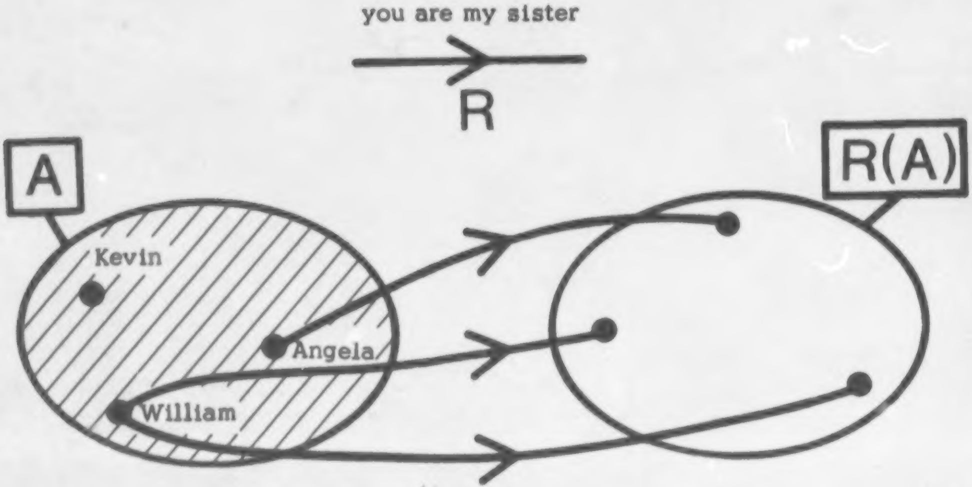
Your picture might look like this one.


you are my sister
R
A
Kevin
Angela
William

T: We have been looking at the relation "you are my sister". (Trace the blue string.) What would the R-image of A be in this case?

S: The sisters of Kevin, Angela, and William.

Ask someone to trace a string for R(A), then draw it yourself and label it "R(A)".


you are my sister
R
A
R(A)
Kevin
Angela
William

L 3

Choose another student. In the dialogue, we will assume the student's name is "Vonda".

T: Where can I put a dot for Vonda? Can I put it in the blue string?

S: No, the blue string is hatched.

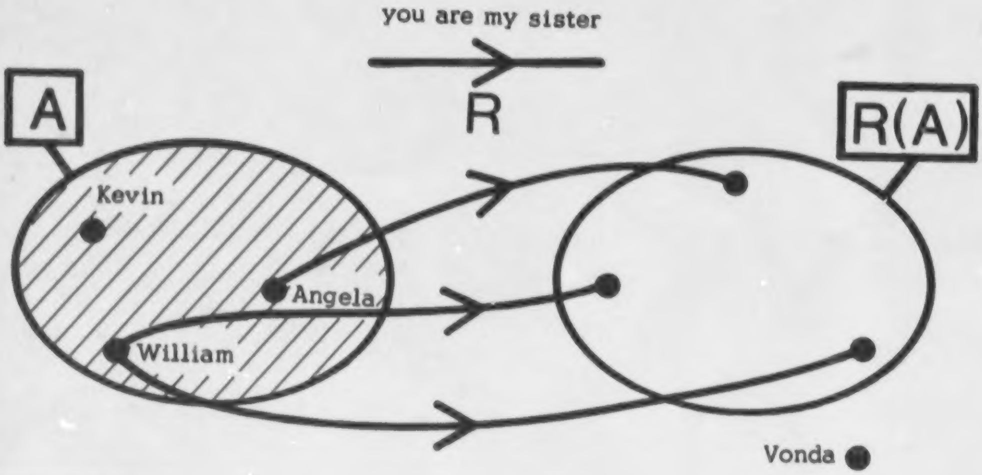
T: . . . in the green string?

S: No, she is not the sister of anyone in the blue string.

T: So where should I put the dot?

S: Outside both strings.

Draw a dot for Vonda.


you are my sister
R
A
R(A)
Kevin
Angela
William
Vonda

T: Vonda, do you have a sister?

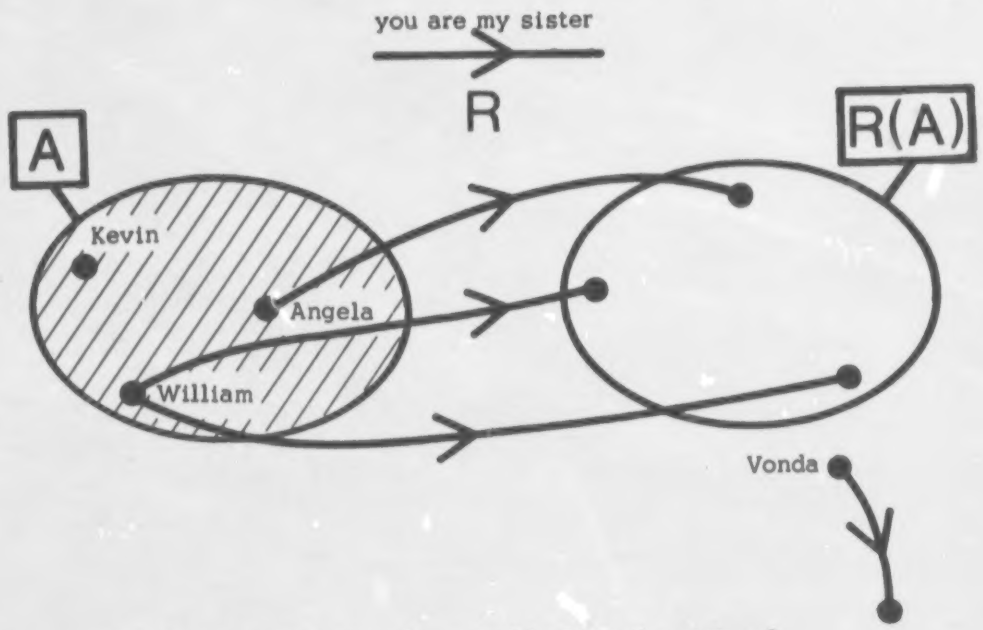
Vonda: Yes, her name is "Michelle".

T: Where should I put a dot for Michelle?

S: Outside the strings.

If someone suggests in the green string, ask if Michelle is the sister of anyone in A.

Draw the dot and the appropriate arrow.

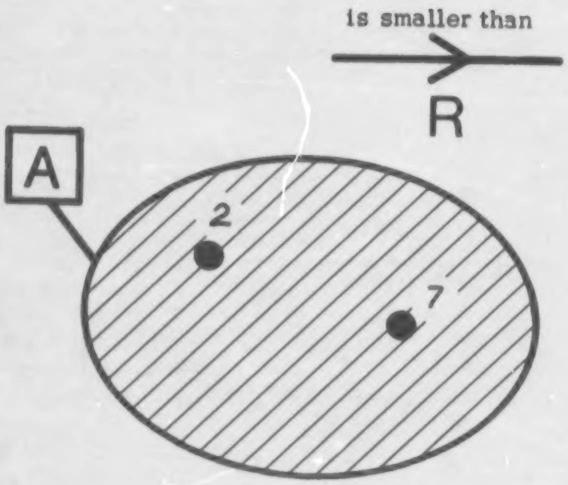


T: Are there any red arrows we could add to the picture?

Let students trace the other red arrows, but do not draw them in the picture. Whenever there is a girl in the blue string, a return arrow should be traced from each of her sisters to that girl. Also, as in our example, a return arrow can be traced from Vonda's sister to Vonda.

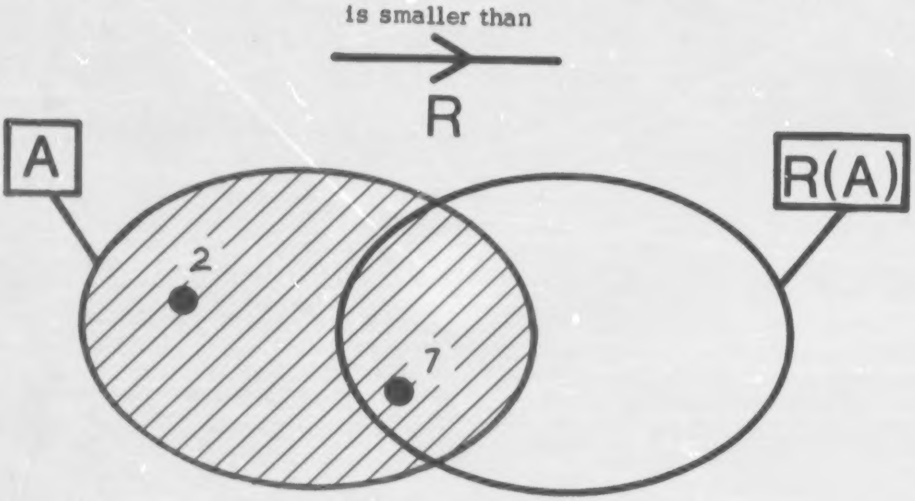
__Exercise 4__ (Optional)

Draw this picture on the board.


is smaller than
R
A
2
7

T: I would like someone to come to the board and trace the string for R(A).

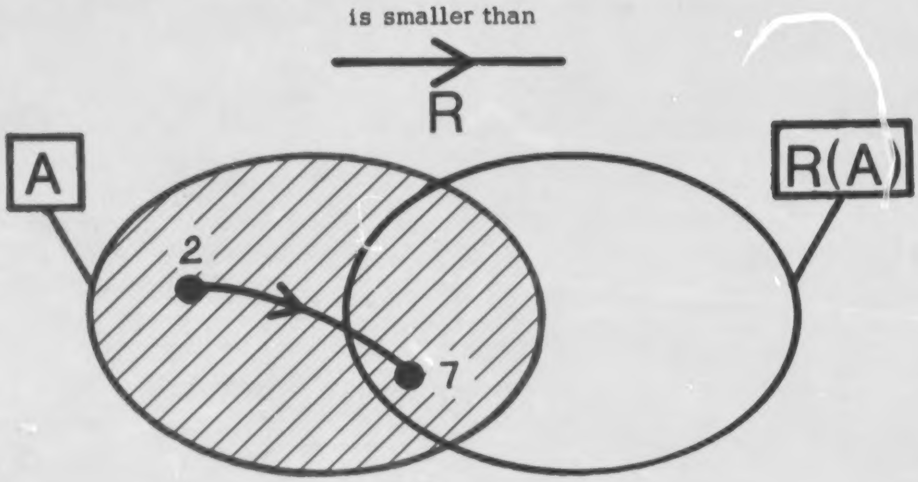
After a student traces the string correctly, draw it and label it "R(A)".


is smaller than
R
A
R(A)
2
7

T: Why must the green string overlap the blue string?

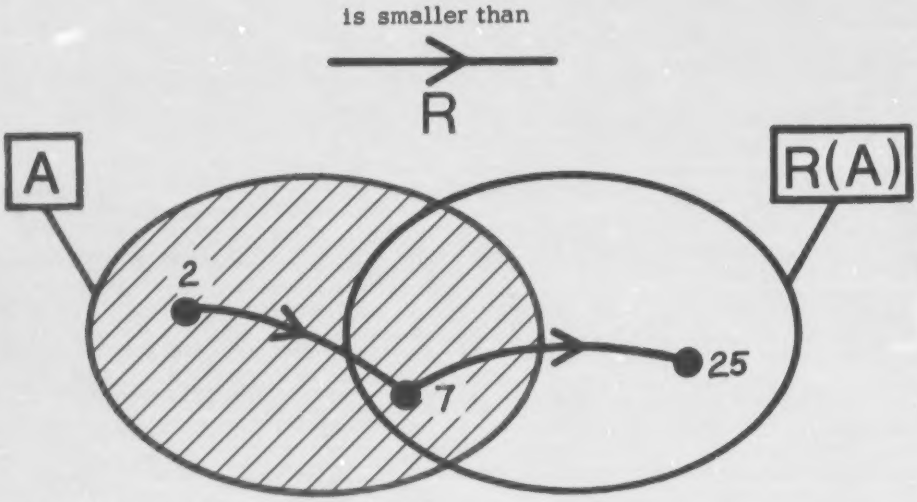
S: Because 2 is smaller than 7, so 7 is in R(A).

Draw the red arrow from 2 to 7.


is smaller than
R
A
R(A)
2
7

T: Who can put the number 25 in the picture?

Let a student put 25 in the picture. Since 25 is in R(A), ask the student to trace an arrow from a number in A to 25. There are two possibilities: the student might trace an arrow from 7 to 25 or an arrow from 2 to 25. It is sufficient to trace just one of these arrows to show 25 is in R(A). Draw only one of the arrows; for example,

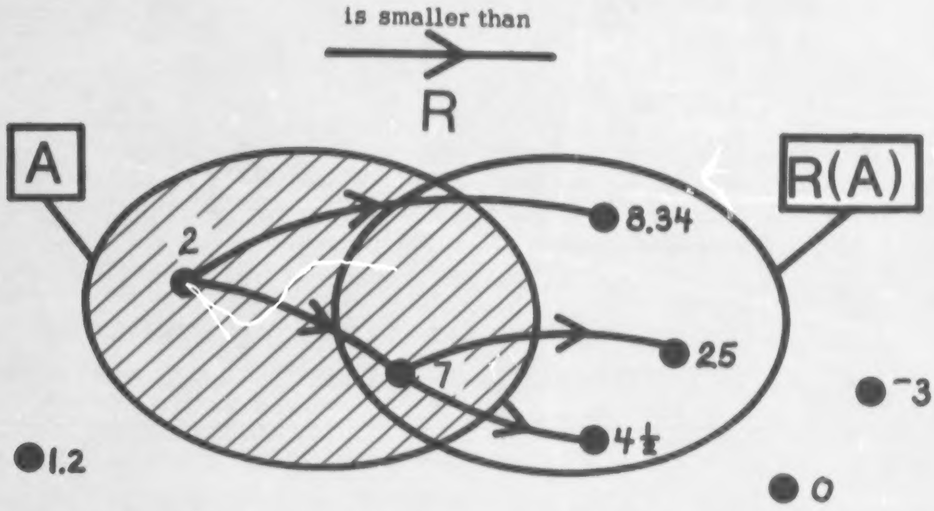

is smaller than
R
A
R(A)
2
7
25

L 3

Ask students to put a variety of other numbers in the picture; some are suggested below and put in the picture in the next illustration.

$$8.34 \qquad {}^{-}3 \qquad 0 \qquad 1.2 \qquad 4\tfrac{1}{2}$$

There are many possibilities for red arrows; only one is drawn in the picture for each dot put in R(A).



T: What numbers can we put in the green string?

S: Numbers larger than 2.

T: Is there a smallest number in R(A)? (No)

If someone says "yes", ask the student which number is the smallest in R(A). Then ask where a number smaller than that one belongs in the picture; for example,

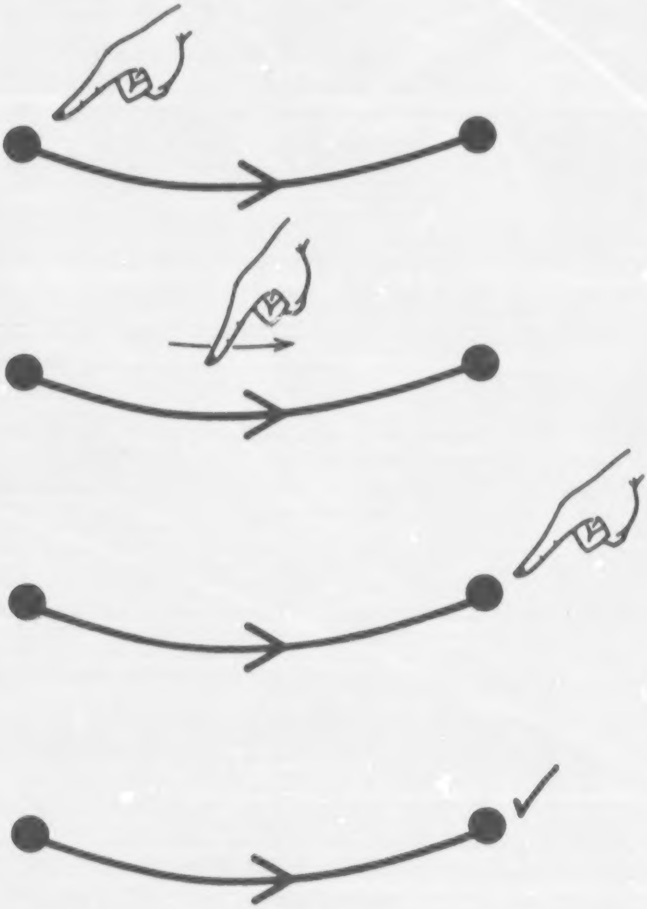
S: 2.1 is the smallest.

T: If 2.1 is the smallest, where does 2.05 belong in the picture?

Since 2.05 has to be in the green string (since 2 is smaller than 2.05), the class should conclude that 2.1 is not the smallest number in R(A). This discussion should make it clear that R(A) has no smallest element.

Worksheet L03 is available for independent work. To students needing help with the worksheet, say this short easily-remembered statement:

Put your finger on a dot in A and check where you end.*



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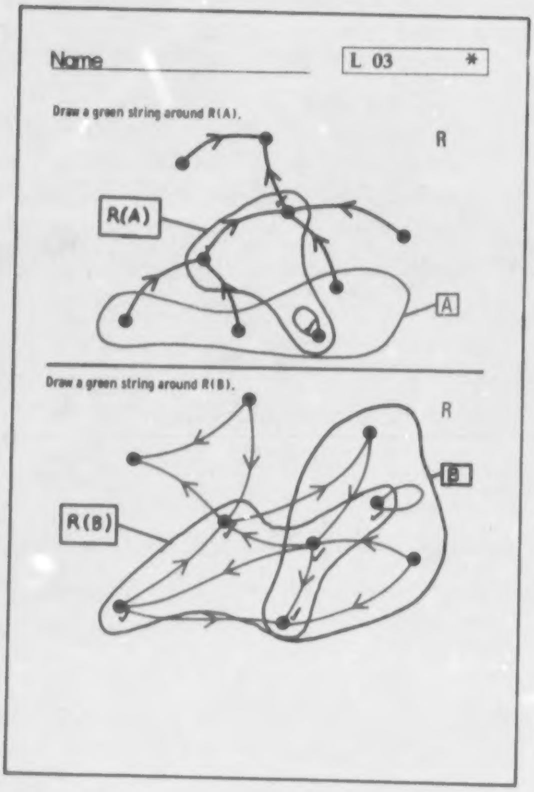
* It is understood that one traces an arrow starting at the dot.



Name ______________________

L 03 *

Draw a green string around R(A).

Draw a green string around R(B).

2.2 Images #2

CAPSULE LESSON SUMMARY

Extend the experiences in Lesson 2.1. Consider an arrow picture where the relation is not identified and one where the relation is +5 and the dots are labeled. Identify the numbers in the image of a finite set of numbers under a numerical relation.

MATERIALS

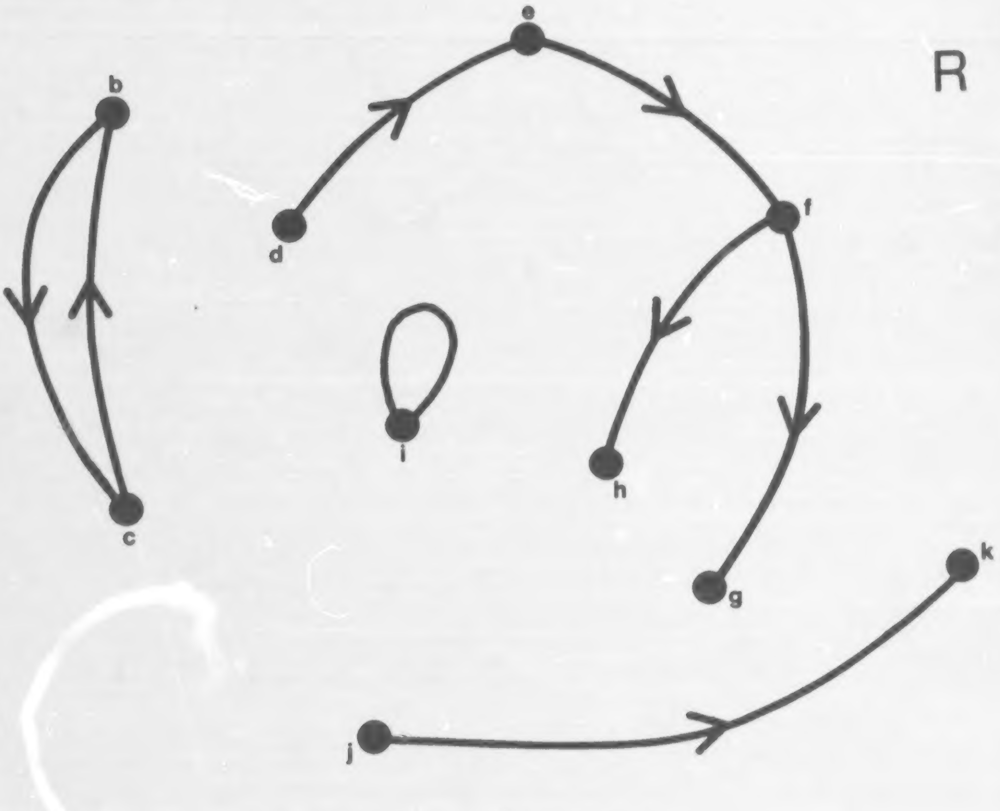
Teacher : Colored chalk ; Language Poster #1; felt tip markers in various colors

Student : Worksheets L04, L05, and L06 ; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Draw this relation carefully on the board.



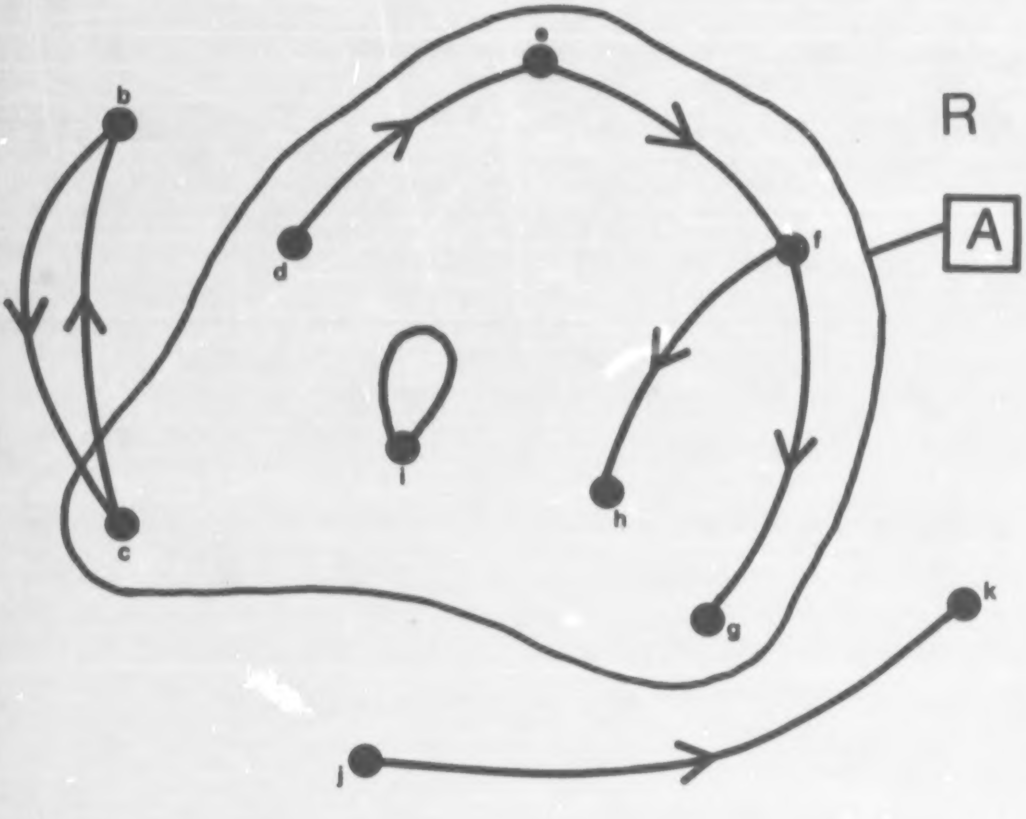
L 7

T: I've drawn an arrow picture for a relation called "R". Is this relation a function?

S: No, two arrows are starting at f, so f is a guilty dot.

Ask a student to trace the two arrows starting at f.

Draw a blue string around only the dots c, d, e, f, g, h, and i, and label the string "A".



T: A is the set of dots in the blue string. What is R(A)?

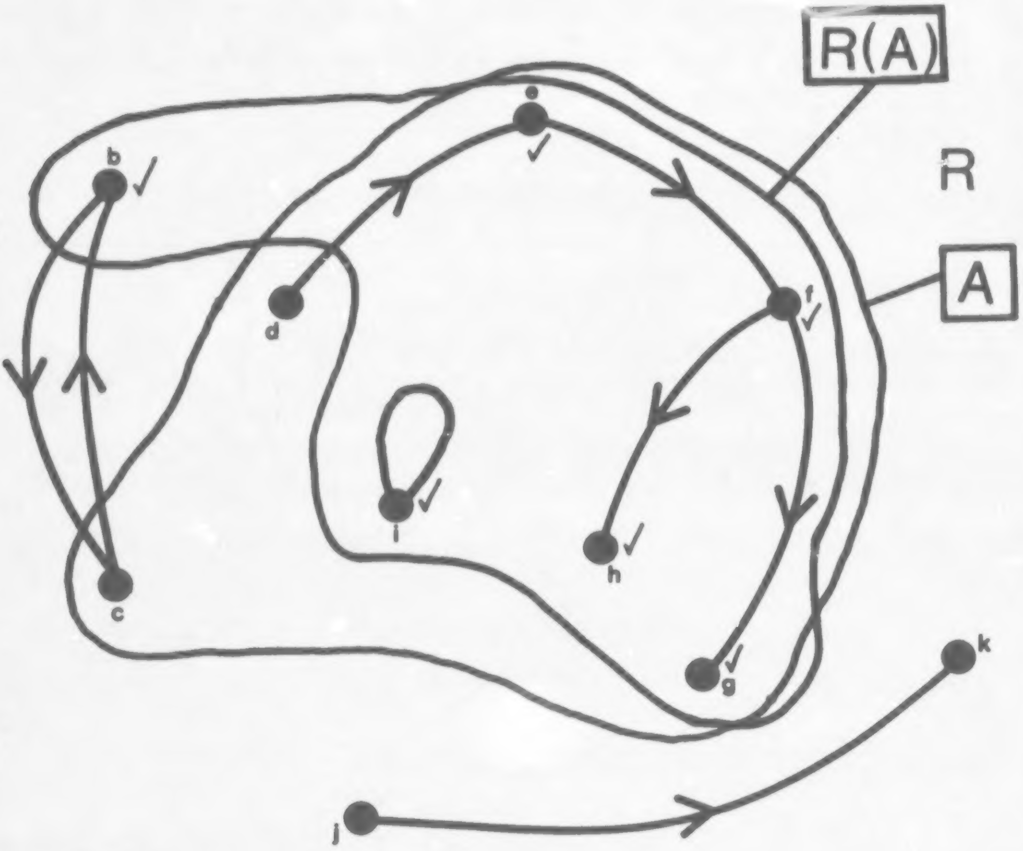
S: The set of ending dots of the arrows which start in A.

T: Who can put a finger on a dot in A, trace a green arrow from that dot, and then check where the arrow ends?

Invite students one by one to trace arrows starting in A and then to draw a red check (√) by their ending dots. Checks should be drawn by b, c, f, h, g, and i.

T: We have red checks by all the dots in R (A). Can someone trace a string around just those dots with red checks?

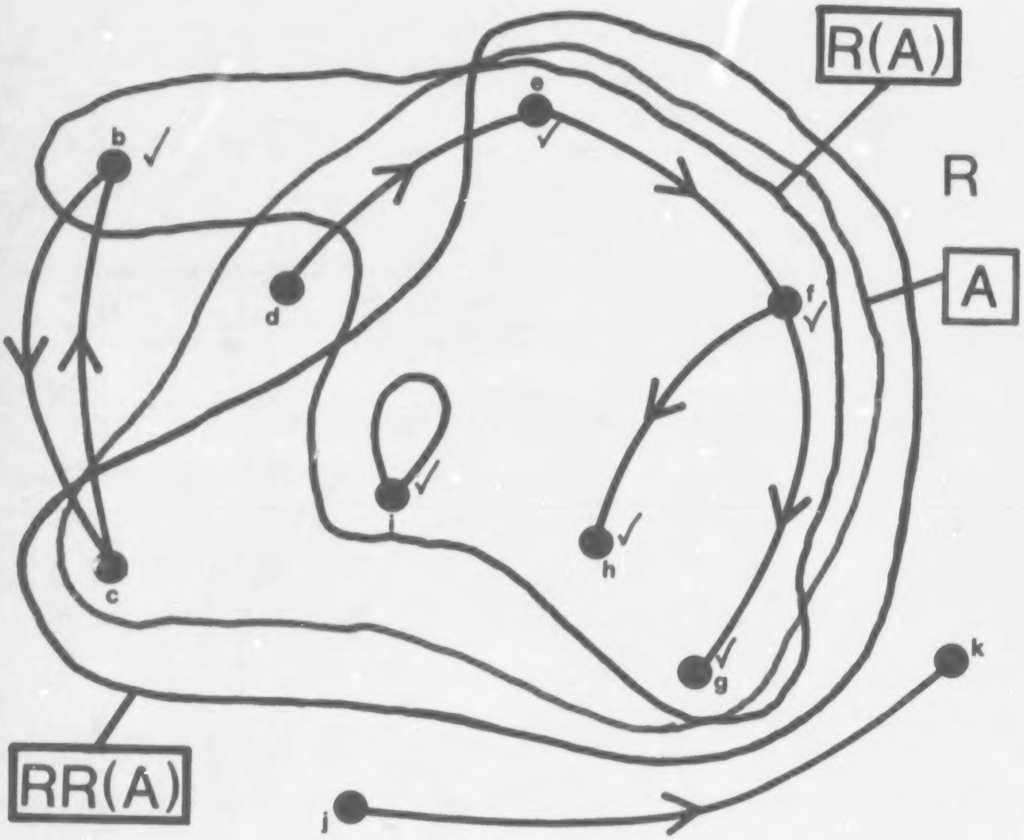
After someone traces the string correctly, draw the string in red and label it "R (A)".



L 7

If you introduced the set RR(A) in the previous lesson in this strand (Exercise 2-A, Lesson 2.1 Images #1), continue with this exercise by asking for the strings for RR(A), RRR(A), and so on. Otherwise, proceed to Exercise 2.

In this picture, the string for RR(A) is in black.



RRR(A) contains b, f, g, h, and i.

RRRR(A) contains c, h, g, and i.

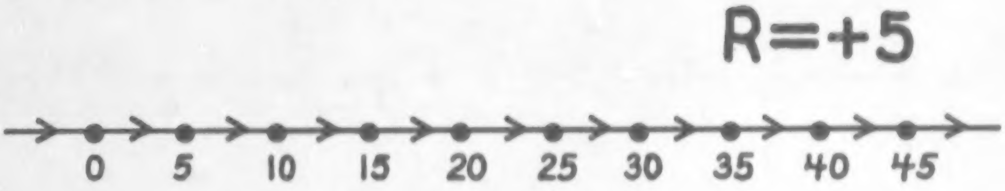
RRRRR(A) contains b and i.

RRRRRR(A) contains c and i.

The successive images after RRRRRR(A) contain i and alternately either b or c.

Exercise 2

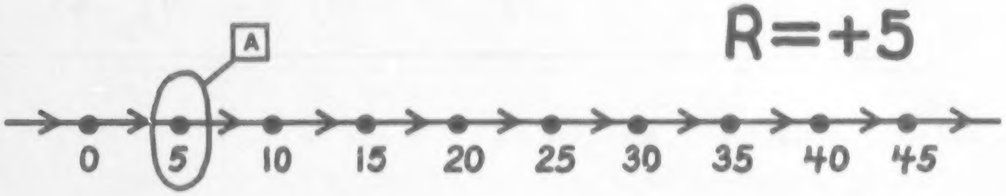
Draw this arrow on the board.


R=+5
0
5
10
15
20
25
30
35
40
45

T: I drew a +5 arrow road and called this relation "R". Is this relation a function?

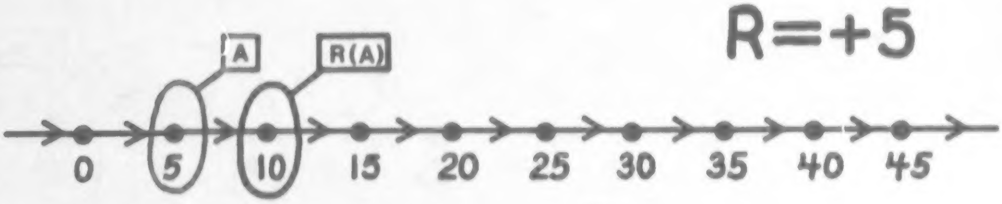
S: Yes, there is only one arrow starting at each dot.

Draw a string around "5" and label it "A".


A
R=+5
0
5
10
15
20
25
30
35
40
45

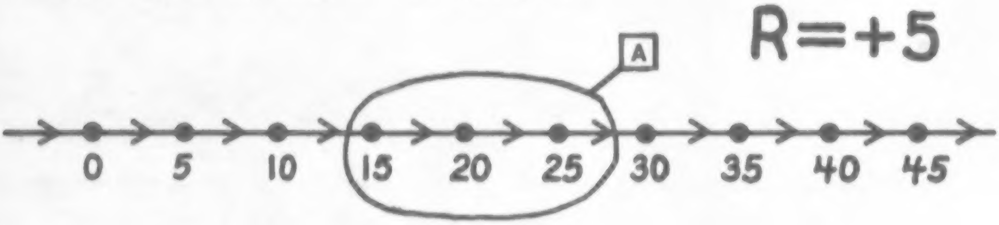
T: A has only one number in it. What is the R-image of A?

Let a student trace the string for R(A); if correct, draw the string yourself and label it.


A
R(A)
R=+5
0
5
10
15
20
25
30
35
40
45

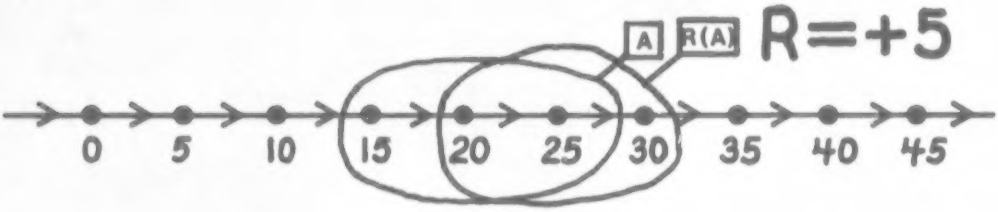
L 7

Erase the string for A and R(A), and then choose a new set of numbers for A and draw a string around them. For example,

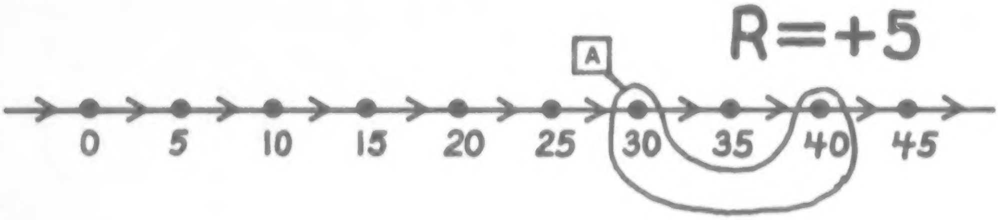

R=+5
A
0
5
10
15
20
25
30
35
40
45

T: A has three numbers in it this time. What is the R-image of A?

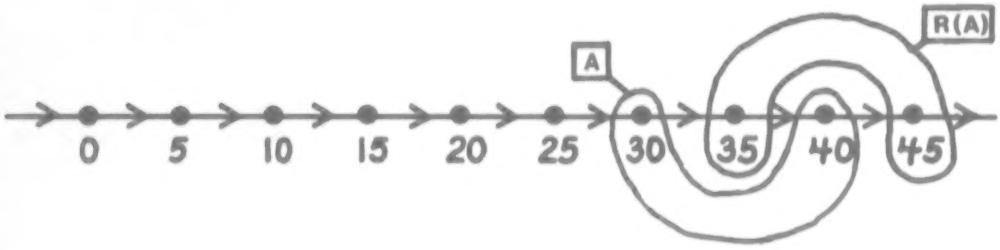
Let a student trace the string for R(A); if correct, draw the string yourself and label it.


A
R(A)
R=+5
0
5
10
15
20
25
30
35
40
45

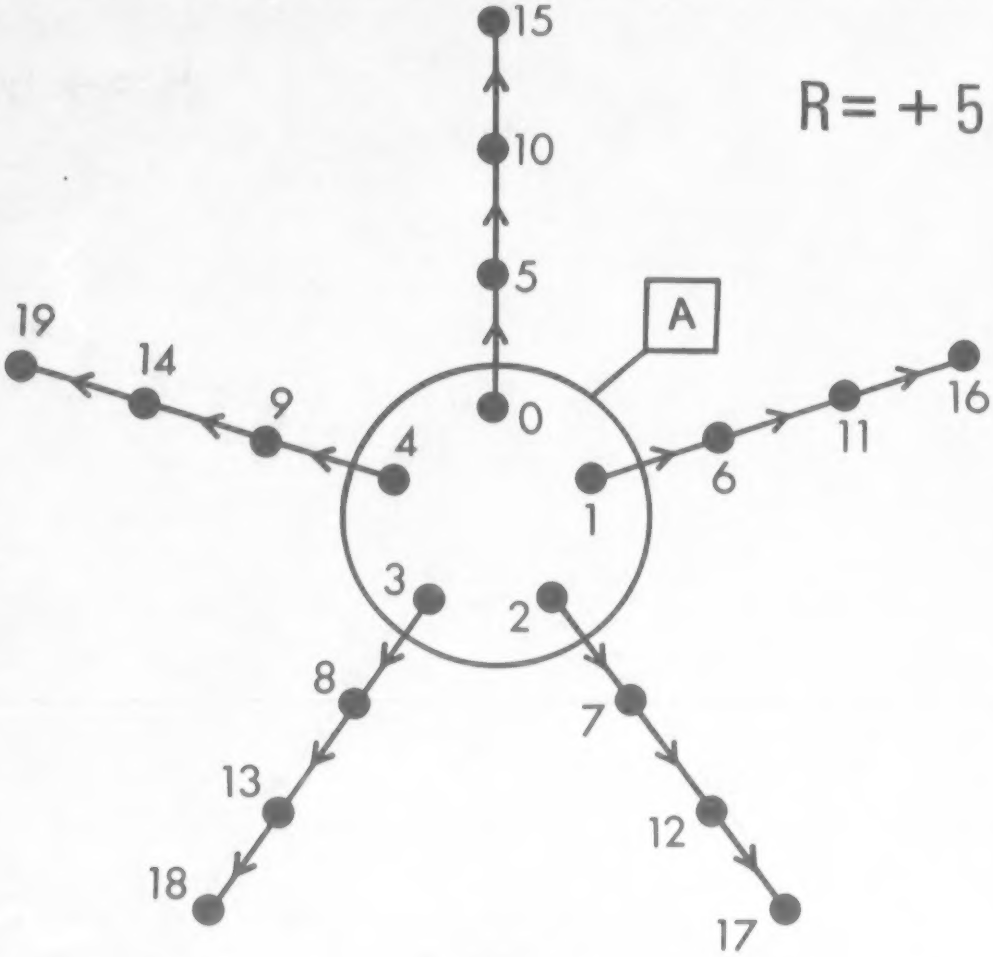
Repeat this activity with another set for A; for example


R=+5
A
0
5
10
15
20
25
30
35
40
45

The R-image of this set is shown below.

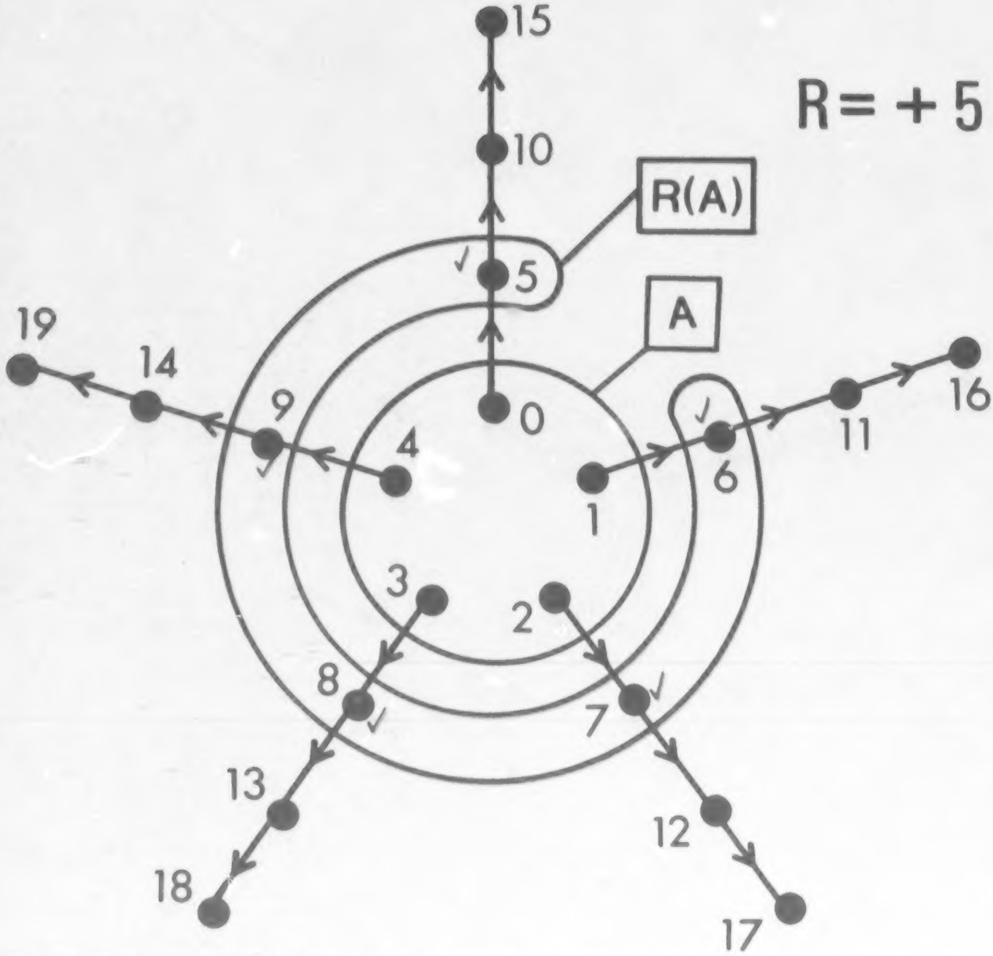

R=+5
R(A)
A
0
5
10
15
20
25
30
35
40
45

Erase the board and tape a copy of Language Poster # 1 for IG-IV to the board.



T: In this picture R is the relation +5. A has five numbers in it. Can someone point to a dot in the R-image of A?

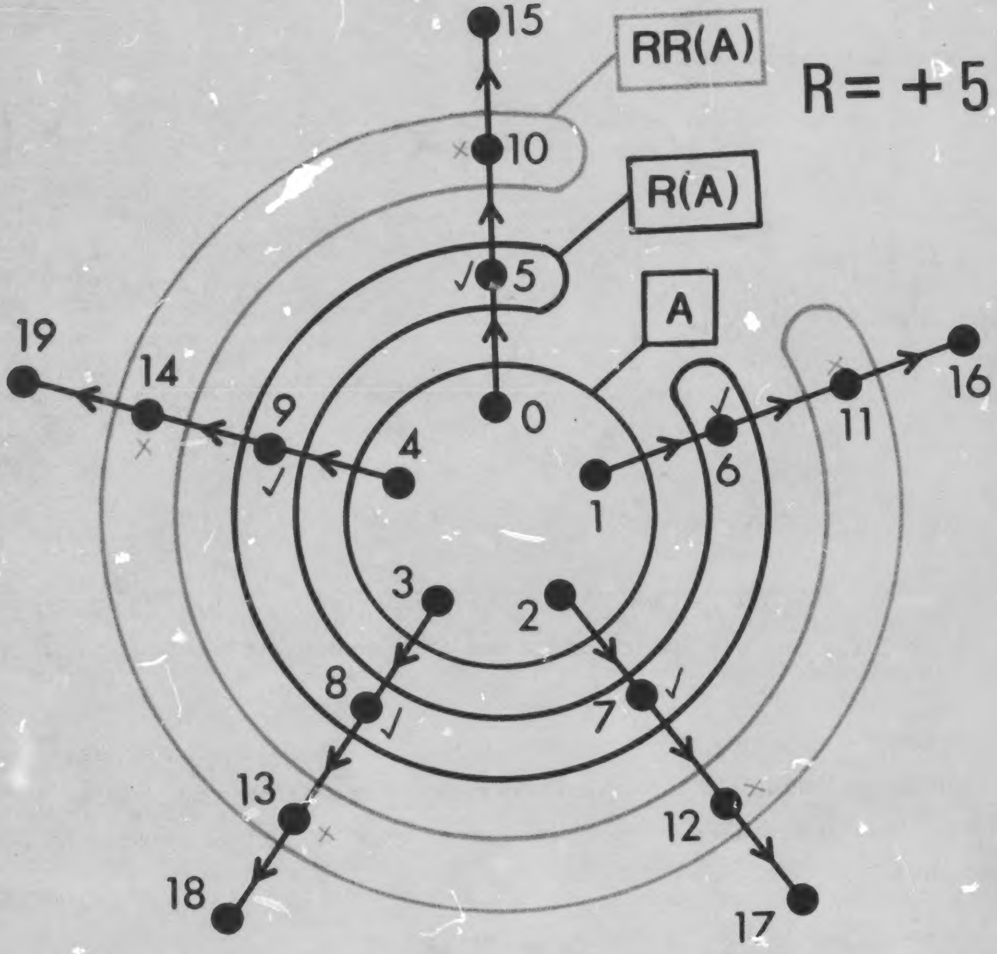
When a student points to a correct dot, check (√) it in green. When all of the dots in R(A) have been found, ask a student to trace a string around just those dots which have a green check by them. If traced correctly, draw the string (in green) as indicated by the student.



NOTE: Students may need help tracing the green string in the preceding illustration so that this string does not include the dots in the blue string. The green string cannot be drawn in the usual way for that reason.

T: Can someone trace the R-image of the green string?

If traced correctly, draw the string (in yellow or another color of your choice) as indicated by the student. If no one traces the string correctly, ask for dots which would be in the R-image of the green string, put a mark (x) by each, and after all of the dots in RR(A) have been found return to the original question. The completed picture (assuming cross marks were drawn) should look like this illustration.


R= +5
RR(A)
R(A)
A
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19

T: What numbers are in the R-image of the yellow string?

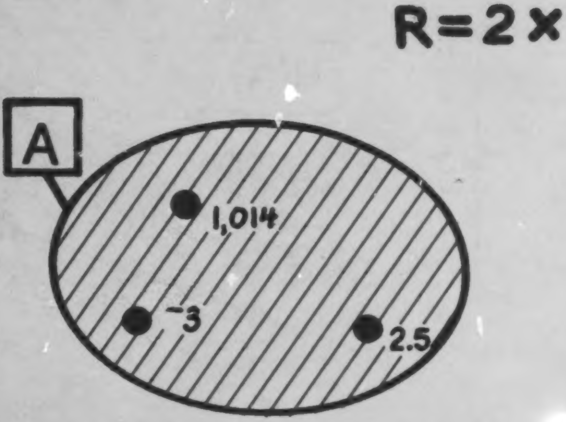
S: 15, 16, 17, 18, and 19.

Take down the poster before beginning Exercise 3.

L 7

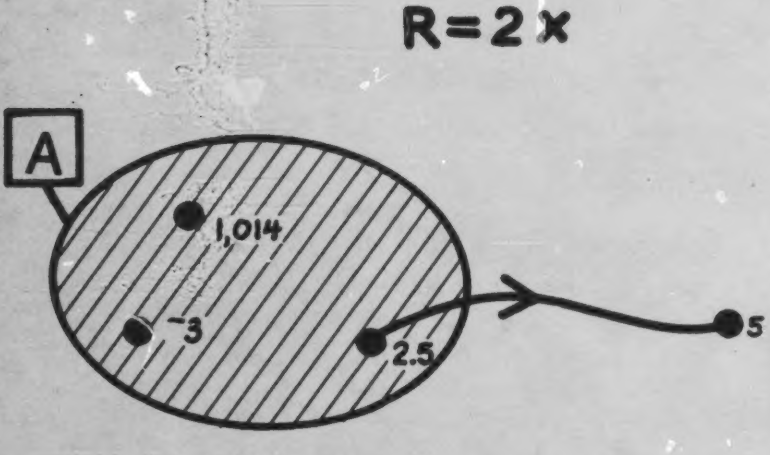
Exercise 3

Draw this picture on the board.

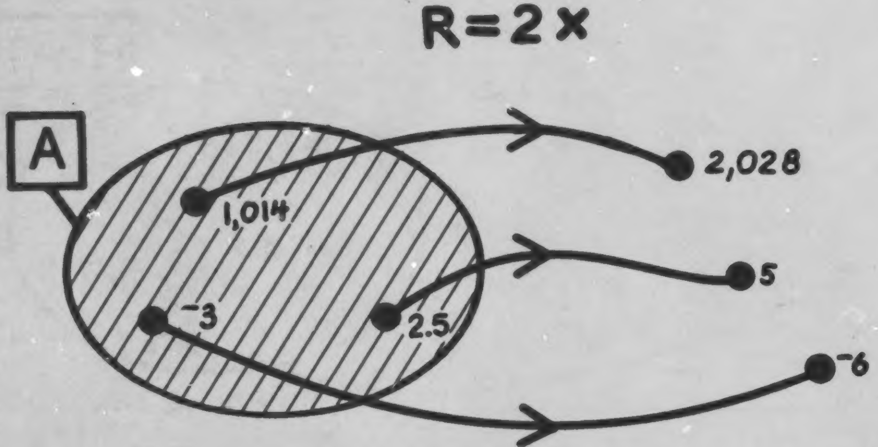


T: R is the relation 2 x. A has three numbers in it: 1,014; $^{-}3$; 2.5. Who can give the R-image of one of these numbers?

Suppose someone gives 5 as the R-image of 2.5. Ask that student to draw a dot on the board for 5 and to draw the appropriate 2 x arrow ending at 5. (The dot for 5 must be outside the blue string because the blue string is hatched.)

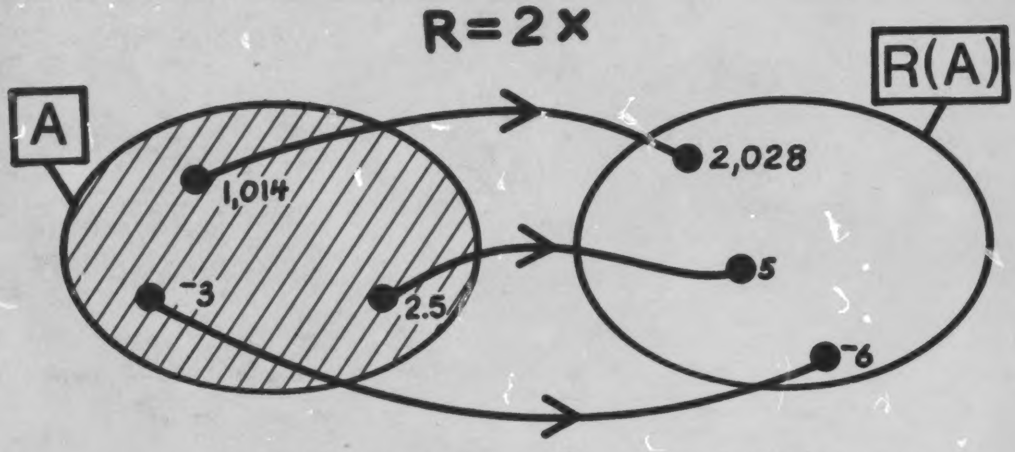


Continue in this way until the R-image of each number in A has been identified.


R=2x
A
1,014
⁻3
2.5
2,028
5
⁻6

T: Could someone draw a string for R(A)?

Let a student draw the string in red.


R=2x
A
R(A)
1,014
⁻3
2.5
2,028
5
⁻6

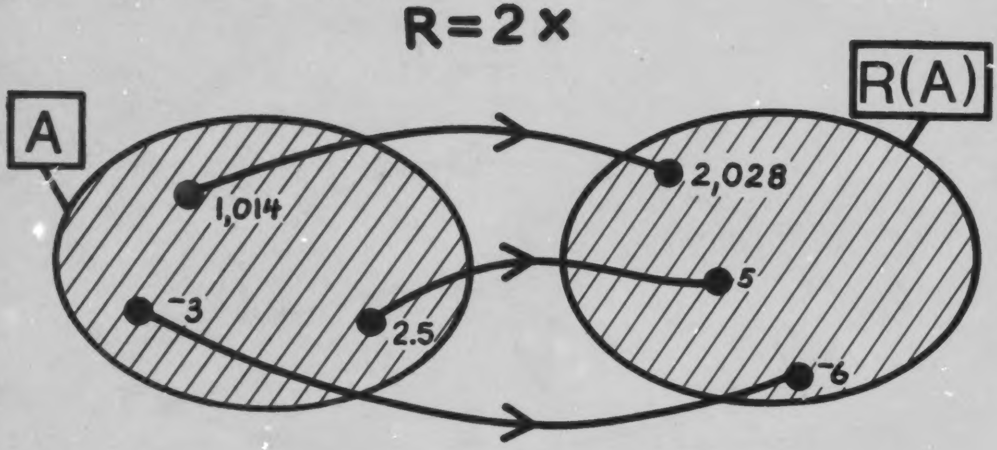
T (tracing the red string): Are there any other numbers in R(A) besides 2,028; 5; and ⁻6?

S: No.

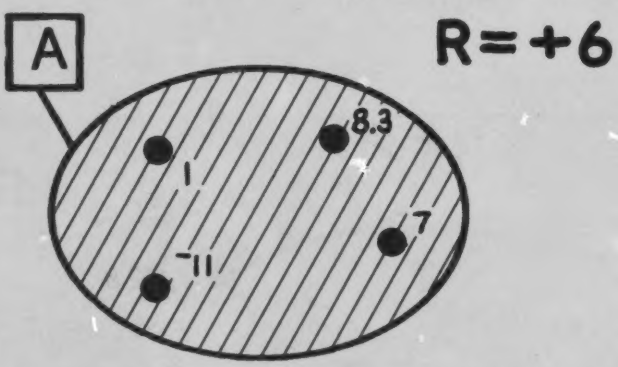
T: How can we indicate that in the picture?

L 7

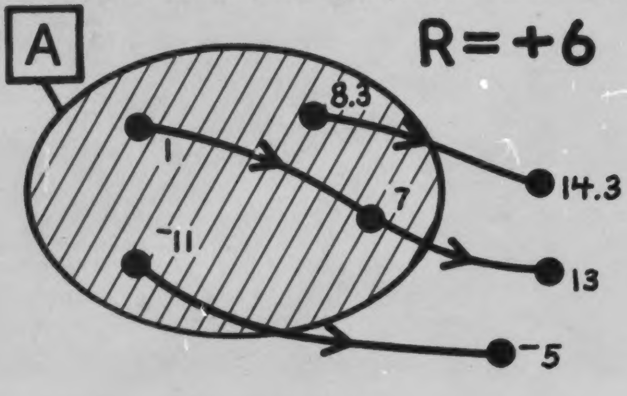
S: Hatch the red string.


R=2x
A
R(A)
1,014
2,028
⁻3
2.5
5
⁻6

Erase the board and then draw this picture on the board.



Ask students for the R-images of the four numbers in A and let them indicate them in the picture by drawing the appropriate dots and green arrows.


A
R=+6
8.3
1
7
⁻11
14.3
13
⁻5

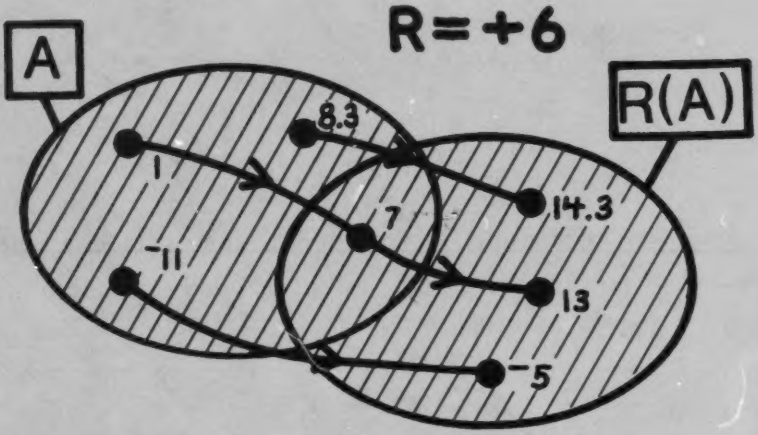
100

T: Which numbers are in R(A)?

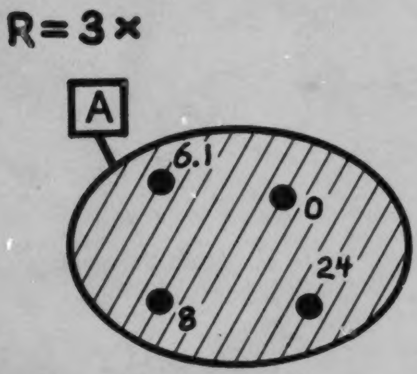
S: 7; 14.3; 13; and 5.

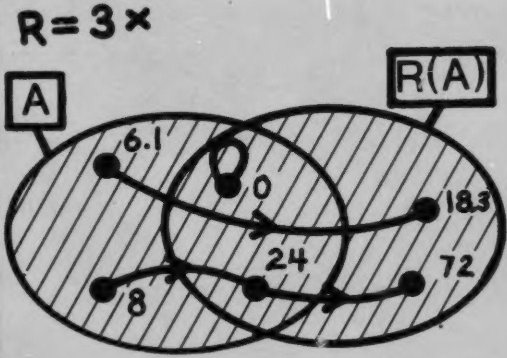
T: Who can draw a string around just those four numbers?

Ask a volunteer to trace the string first so that you can make sure 7 is included in R(A). After the string is drawn, ask if the string for R(A) can be hatched. (Yes)


R=+6
A
R(A)
8.3
1
7
-11
14.3
13
-5

Repeat this activity with a new situation such as the one pictured below. The correct solution is given on the right. Use numbers appropriate, but not too easy, for your students.

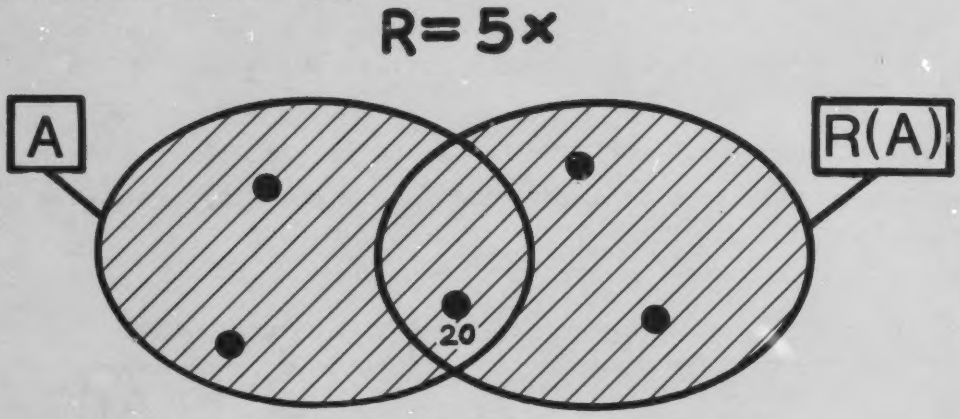



R=3×
A
R(A)
6.1
0
18.3
24
72
8

L 7

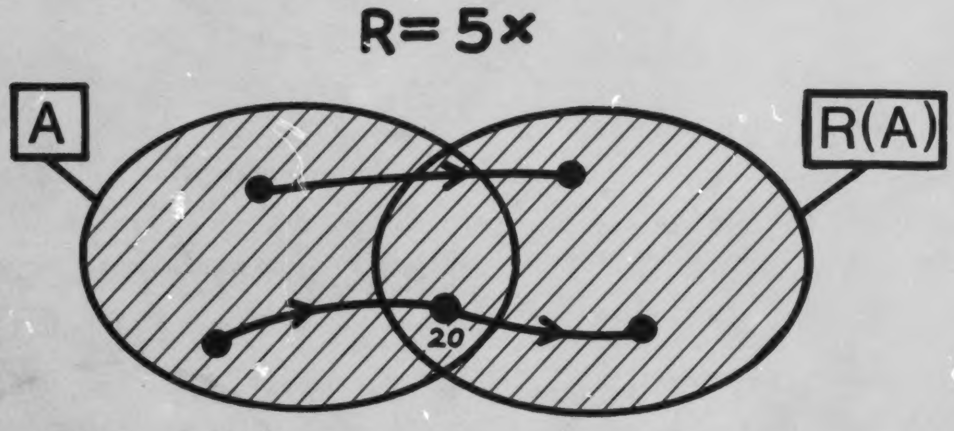
T: Now R is 5 x. This time I'm going to draw A and R (A).

Draw this picture on the board.

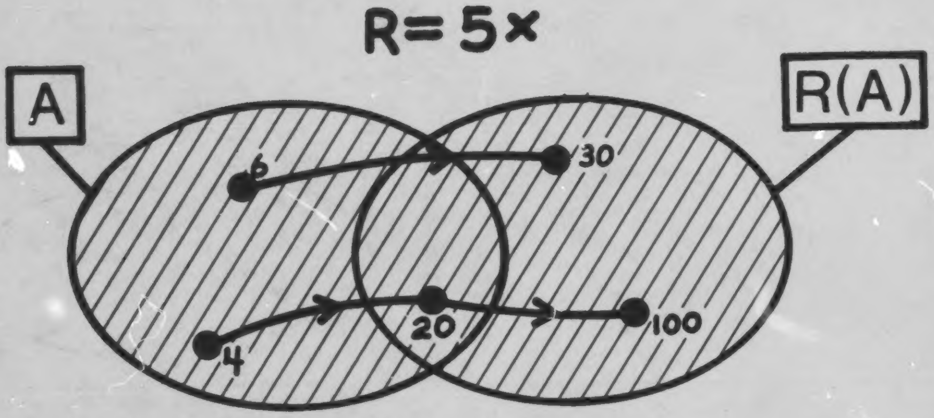


T: Could someone draw a green arrow in the picture?

Continue asking for green arrows until three of them have been drawn in the picture: one ending at 20, one starting at 20, and one connecting the remaining two dots but starting at the dot in A.

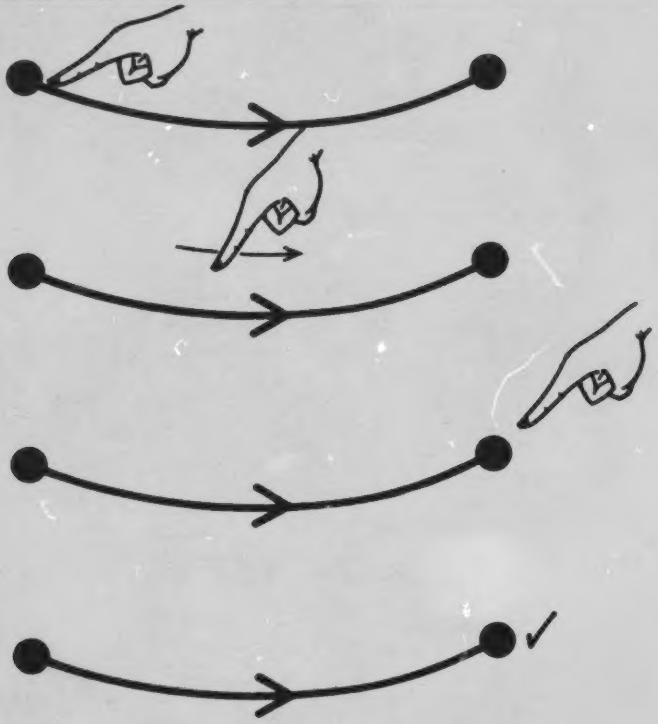


Ask students to label the dots in the picture. Indicate there are several possibilities for two of the dots; one such possibility is shown in the next illustration.



Worksheets L04, L05, and L06 are available for independent work. Students having difficulty with Worksheet L04 might be helped with this short easily-remembered statement:

> Put your finger on a dot in A and check where you end.*



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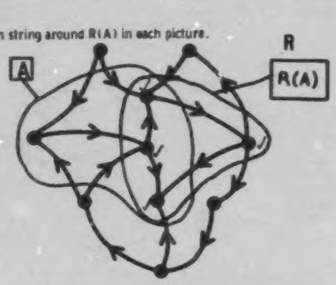
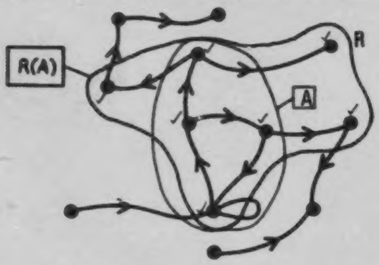
* It is understood that one traces an arrow starting at the dot.

Name ____________________ L 04 *

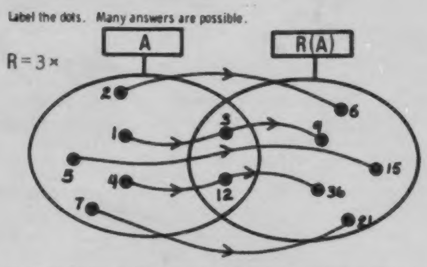
Draw a green string around R(A) in each picture.



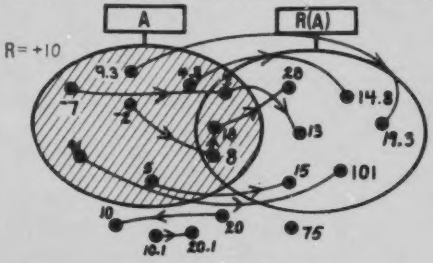


Name ____________________ L 05 **

Label the dots. Many answers are possible.


A
R(A)
R = 3×
2
6
1
3
9
5
15
4
12
36
7
21

Draw more blue arrows and then label the dots. Many answers are possible.

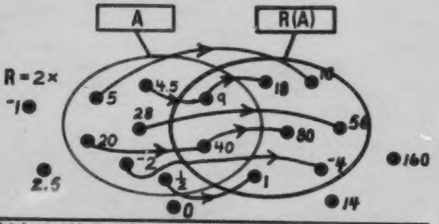

A
R(A)
R = +10
9.3
⁻7
⁻2
20
14.8
13
19.3
14
8
15
101
10
20
75
10.1
20.1

(Many solutions are possible.)

Name ____________________ L 06 ***

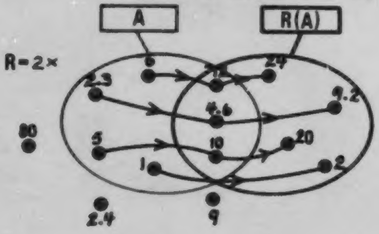
Label each dot in the string picture with one of these numbers.

56 1 ⁻1 10 ⁻4 2.5 18 0 14 80 160


A
R(A)
R = 2×
⁻1
5
4.5
9
18
10
28
56
20
80
40
⁻2
⁻4
160
2.5
1
0
14

Label each dot in the string picture with one of these numbers.

4.6 ⁻24 80 5 ⁻12 9.2 2 2.4 20 9 1 2.3 ⁻6 10


A
R(A)
R = 2×
6
24
2.3
9.2
4.6
80
5
10
20
1
2
2.4
9

2.3 Guess My Rule #1

CAPSULE LESSON SUMMARY

Play a Guess My Rule game where the rule is for a function R on the set of points in the plane. Apply this rule to find the R-images of a variety of sets of points. Given an set, determine some possibilities for a set of points with the given set as its R-image.

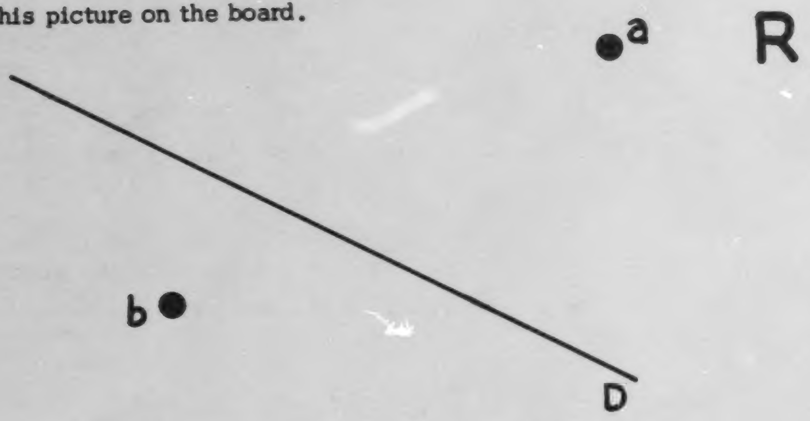
MATERIALS

Teacher: Colored chalk; meter stick or other straightedge
Student: Colored pencils; Worksheets L07, L08; unlined paper

DESCRIPTION OF LESSON

Exercise 1

Draw this picture on the board.



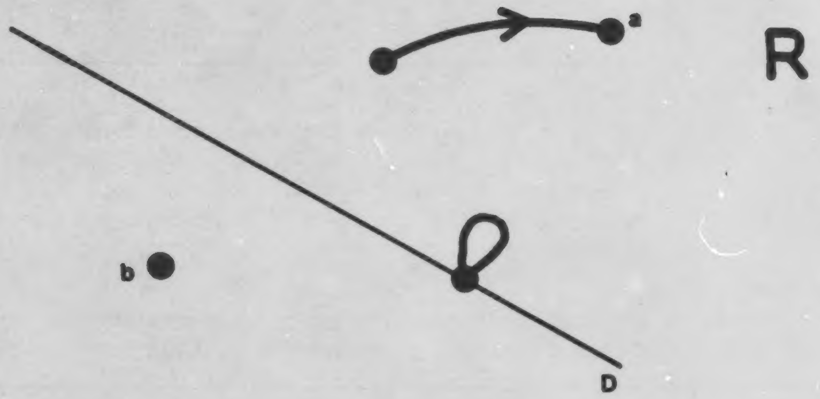
T: Today we will play a Guess My Rule game, but this time my rule is not a rule about numbers. Instead, it is a rule about points and their images. Here are two special points, a and b, and a line D. (Point to "R".) My rule is a function that I will call "R". Do you remember what a function is?

S: At most one green arrow starts at each point.

T: Yes. And my rule is a special function, because each point is the starting point of exactly one green arrow. Here are two clues.

L 9

Add this information to your picture.

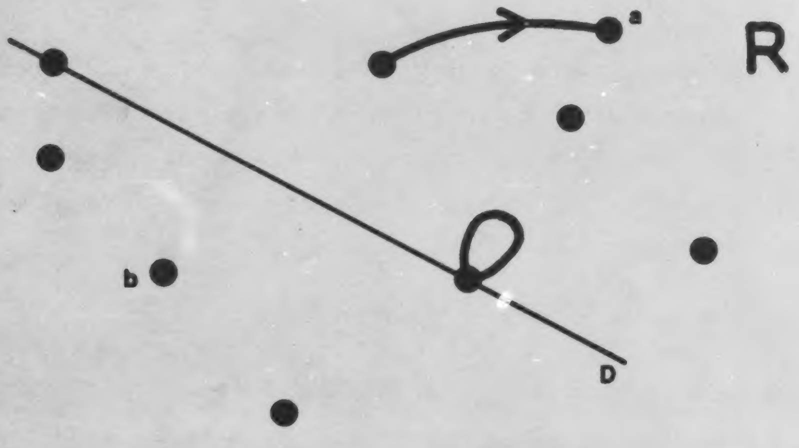


NOTE: The rule to be discovered is:

- a is the image of every point on the same side of D as a;
- b is the image of every point on the same side of D as b;
- each point of D is its own image.

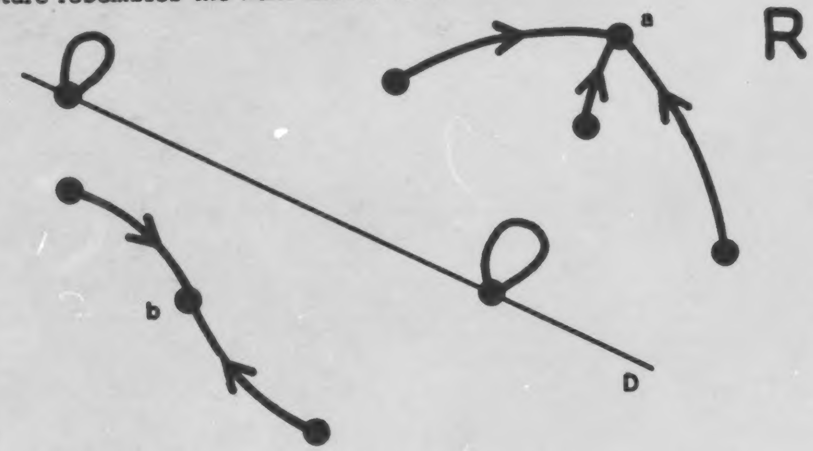
Do not announce the rule but tell the class to think about what the rule could be. Ask the students not to reveal the rule when they think that they have guessed it.

One at a time, draw the six additional dots shown in the next illustration. After each addition, ask the class to guess how to draw the green arrow starting at that dot.



106

When a student correctly guesses the image of a point, draw the appropriate green arrow or loop. If no one guesses the correct image of a dot, draw an arrow or loop starting at that dot as another clue. Continue in this way until your picture resembles the next illustration.



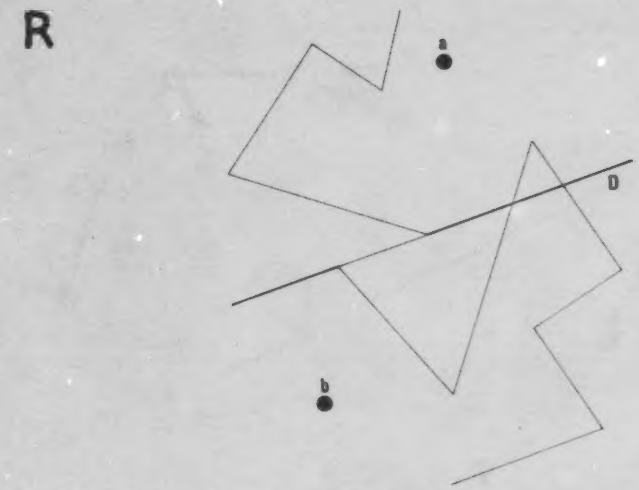
After these dots and arrows have been added to the picture, ask someone to explain your rule. If no one can do so, continue the discussion by adding a few more dots and the arrows starting at these dots until a correct guess is forthcoming.

Choose several students, one at a time, to play Follow My Finger with the picture for R. You point to a dot and the student playing with you points to the image dot. After having used the dots whose arrows are already in the picture, choose some others. Of course, the rule requires that when your finger is on the same side of D as a, the student points to a and when your finger is on the same side of D as b, the student points to b. If your finger is on the line, the student must indicate the same point as you. In other words, as you move your finger continuously over the "a-side" of D, the student should constantly point to a. But when you cross D, the student should move quickly to the crossing point with you and then to b as you enter the "b-side" of the line. Playing Follow My Finger will reinforce the rule. Encourage the class to watch carefully for mistakes.

L 9

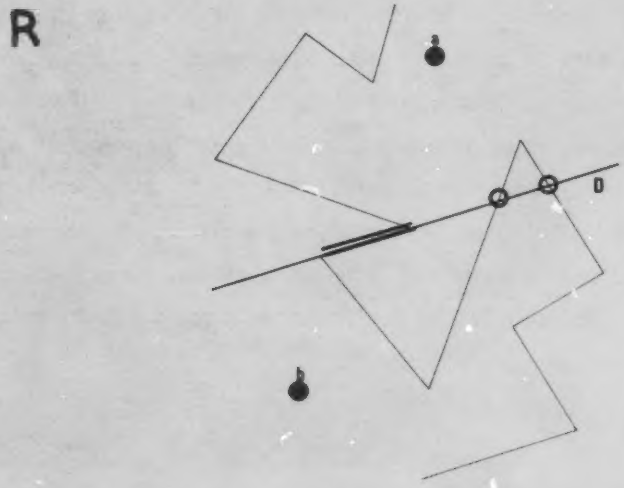
Exercise 2

Draw this picture on the board.



T: Here is a blue zigzag. The rule of R is the same as before. Let's find the R-image of the zigzag. We will draw the R-image in red.

Start by asking students to find the images of just a few points of the zigzag in your board drawing. The full image is shown in red in the next illustration. Then distribute Worksheet L07, which pictures a similar zigzag. Instruct the students to find the R-image of the zigzag and to color it red. When most students have finished, complete the board drawing and play Follow My Finger with two or three students.

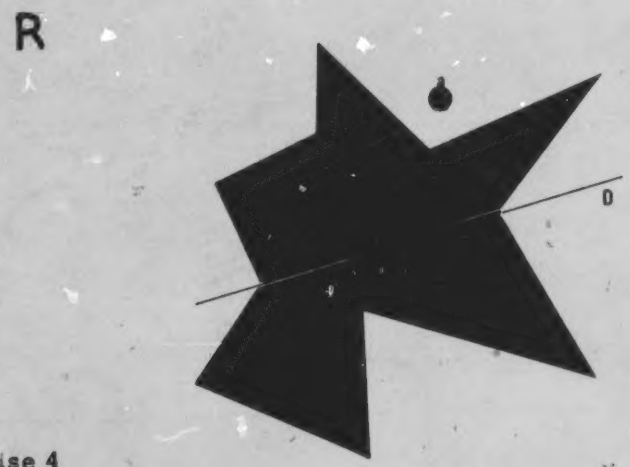


Exercise 3

Distribute Worksheet L08.

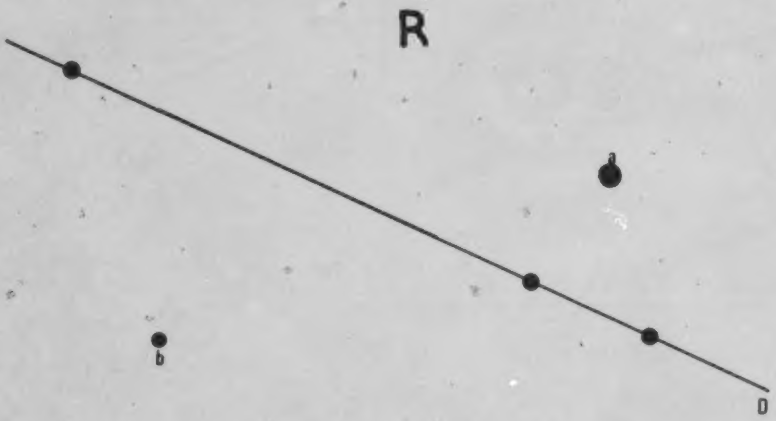
T: Here is another problem. This time you must find the R-image of a blue region. Draw the R-image in red.

When most students have finished, collectively check the solution and then continue to the next exercise. The key for this worksheet is shown in the next illustration.



Exercise 4

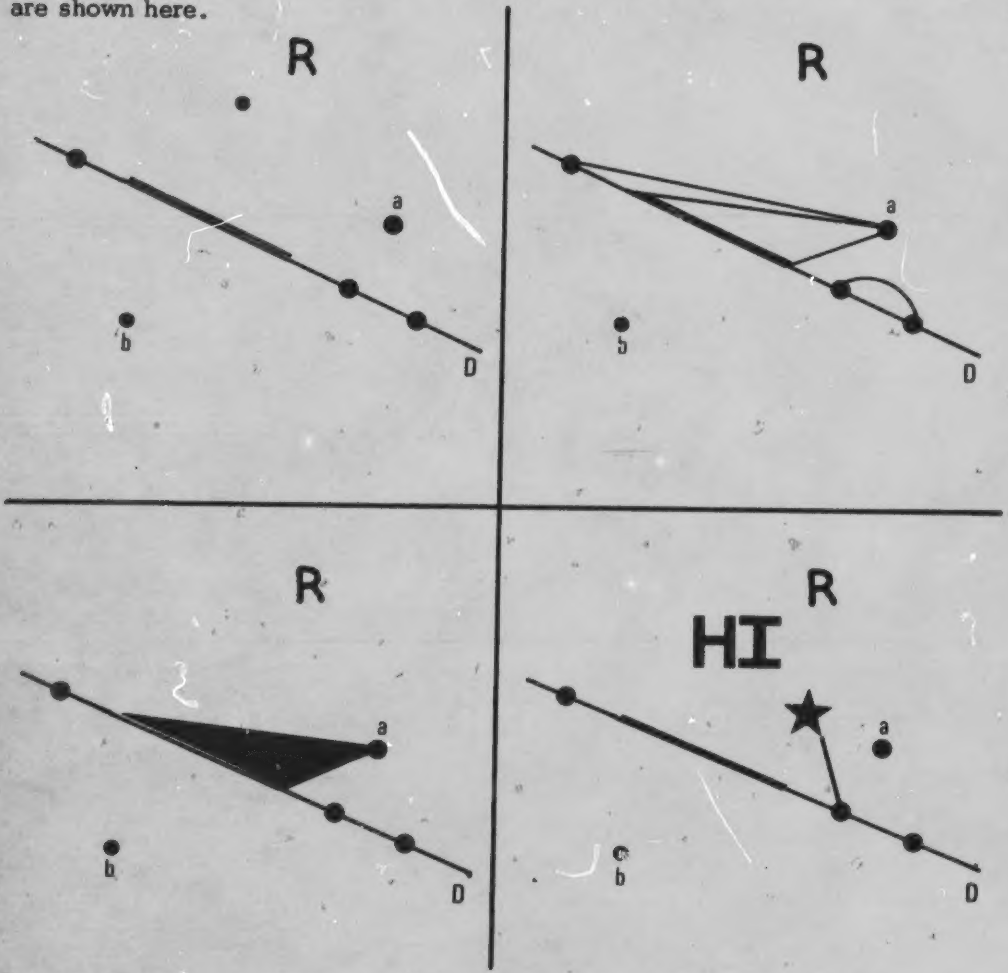
Distribute unlined paper and draw this picture on the board.



L9

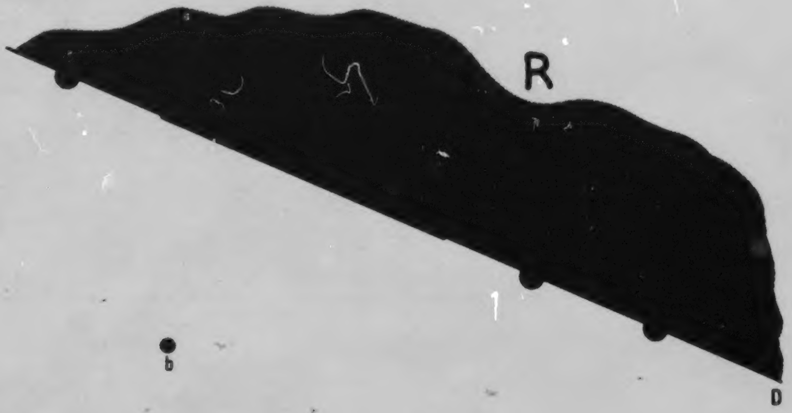
T: On your paper, draw a line D and points a and b as I have done. Then color points red as I have done here. The red points are all of the points in the R-image of some set of points. Color blue the points in a set whose R-image is this set of red points.

Let the students work independently on this problem. If necessary, remind students as they work that b is not in the image. After a short time, invite a student to draw a solution on the board. Four of the many possible solutions are shown here.



T: Turn over your paper and draw a line D and points a and b again. Color the same points red as before. Now try to find the largest possible set of points that has this set of red points as its R-image. Color the points in the set blue.

When several students think they have found a solution, lead a discussion to interpret it. If there is some difficulty in finding the solution, invite several students to the board to show their solutions. Since each will have a different idea of the solution, the combination of their solutions will approach the full solution and the class should soon realize that the set illustrated below is the solution.

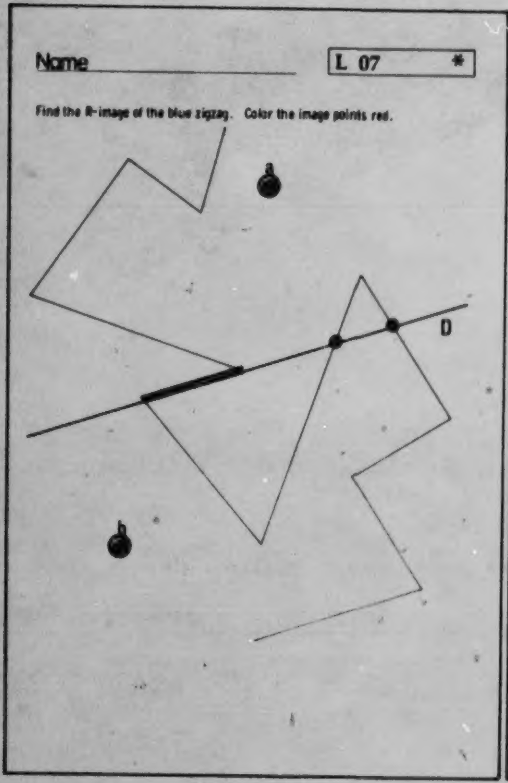


NOTE: The class discussion of this situation should consider these observations:

- Since b is not red, no points on the "b-side" of D are blue.
- Since a is red, all points on the "a-side" of D are blue.
- Since the R-image of a point of D is the point itself, a point of D is blue if and only if it is red.

Emphasize the last of these observations by successively indicating points on D and asking whether it should be colored blue or not.

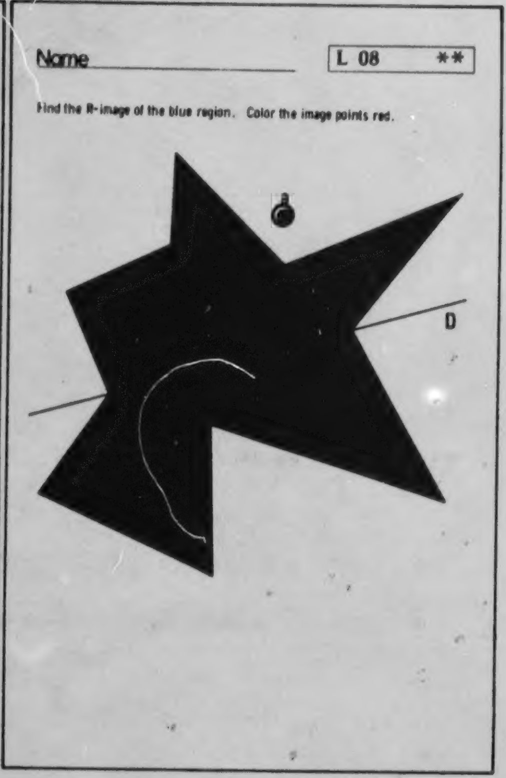
If time remains, ask students to find other (smaller) solutions to the original problem of this exercise. Invite students to draw solutions on the board for discussion.



Name

L 07 *

Find the R-image of the blue zigzag. Color the image points red.



Name

L 08 **

Find the R-image of the blue region. Color the image points red.

2.4 Images #3

CAPSULE LESSON SUMMARY

Examine the composition of a 2 x relation with itself one or more times. Find R(W), RR(W), RRR(W), . . . , where W is the set of whole numbers and where R is the relation 2 x.

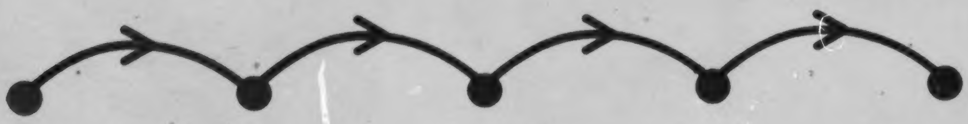
MATERIALS

Teacher : Colored chalk
Student : Worksheets L09, L10, L11

DESCRIPTION OF LESSON

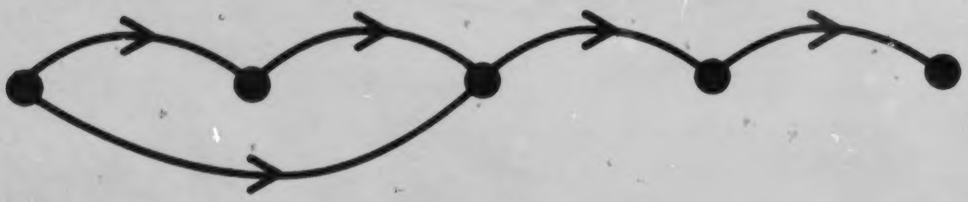
Exercise 1

Draw this arrow picture on one side of the chalkboard.



Do some mental arithmetic by letting a student choose a number for one of the dots and then asking the class what the other numbers in the picture would be. Repeat this activity several times. The arithmetic could become difficult if, for example, an odd number is chosen for one of the dots to the right or if a number such as 738 is chosen for one of the dots to the far left.

Draw this blue arrow in the picture.



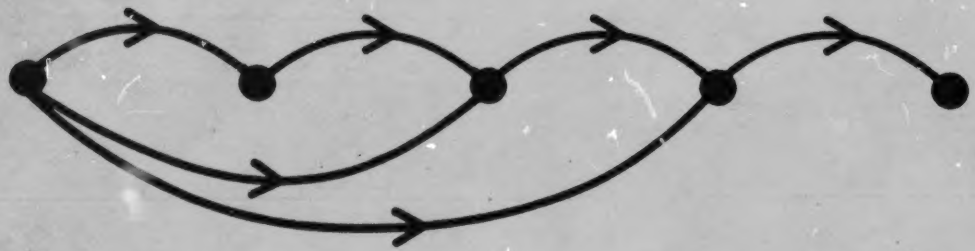
L 11

T: What could the blue arrow be for?

S: 4 x.

Write " 4x " under " 2 x ". Draw this green arrow in the picture.

2x
4x

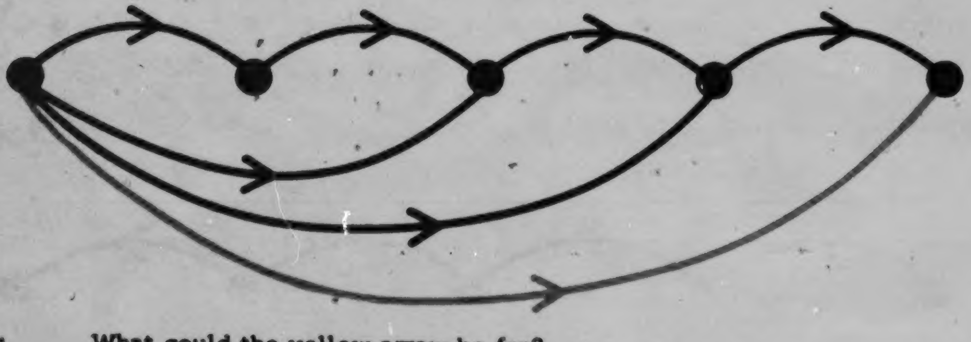


T: What could the green arrow be for?

S: 8 x.

Write " 8x " under " 4 x ". Draw this yellow arrow in the picture.

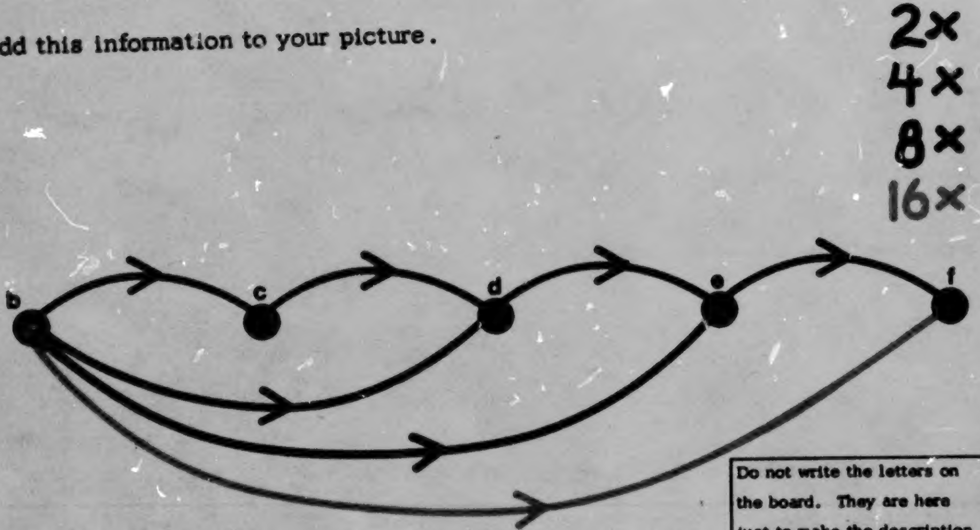
2x
4x
8x



T: What could the yellow arrow be for?

S: 16 x.

Add this information to your picture.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: If this dot (point to b) is for some whole number, what can we say about the number which is here (point to c)?

S: It is a whole number.

S: It is an even number (a multiple of 2).

T: . . . and the number which is here (point to d)?

S: It is a whole number.

S: It is a multiple of 4.

T: Why is it a multiple of 4?

S: Because it is the ending dot of a 4 x arrow.

Trace the blue (4 x) arrow.

T: . . . and the number which is here (point to f)?

S: It is a whole number.

S: It is a multiple of 16, because it is the ending dot of a 16x arrow.

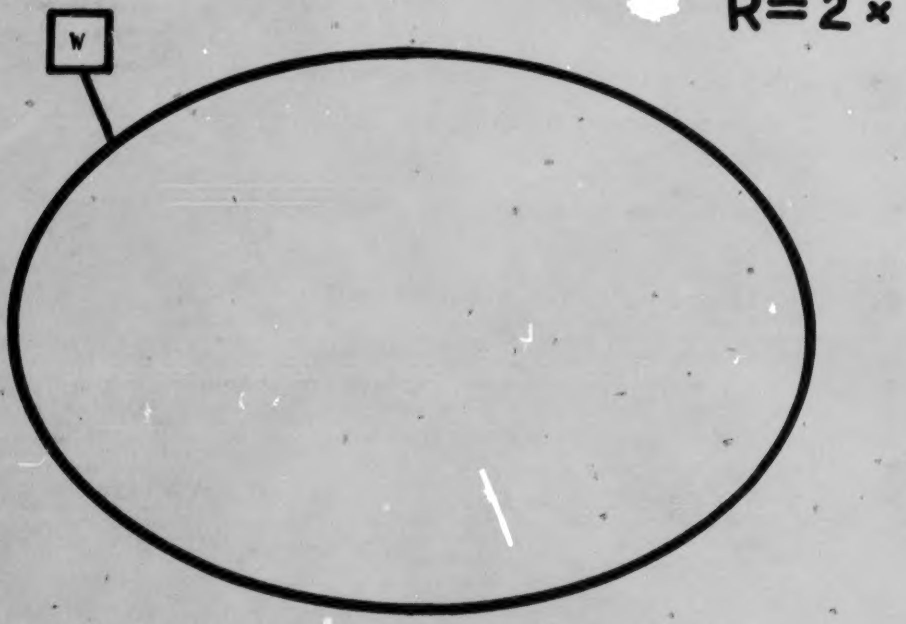
Trace the yellow 16x arrow.

Keep the picture on the board for use in Exercise 2.

Exercise 2

T: Let's look at this idea in another way.

Draw a large string on the board and label it "W". Indicate that the relation R (in red) is 2x.



T: This string is for the whole numbers. Which numbers are whole numbers?

S: 0, 1, 2, 3, ...

T: Is $\frac{1}{2}$ a whole number? (No) Is 1.7 a whole number? (No) Is $^{-}8$ a whole number? (No)

T: The relation R is 2x. What number is the R-image of 30? (60) . . . of 0? (0) . . . of 1? (2) . . . of 2? (4)
Give me another whole number and its R-image.

S: The R-image of 7 is 14.

Let several students give the R-image of whole numbers of their choice.

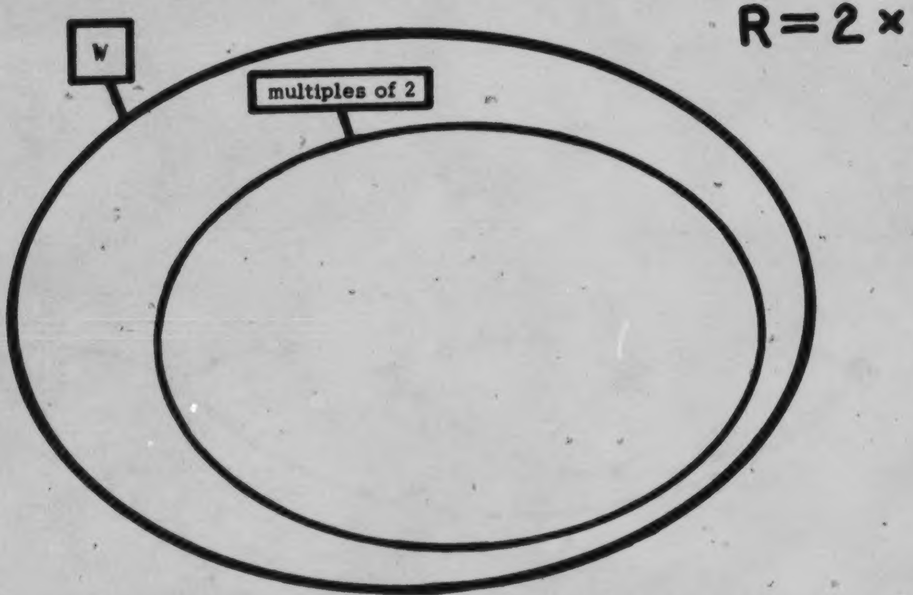
T: What can we say about the numbers in the R-image of W?

S: They are even numbers; all the whole numbers which are even.

S: They are all multiples of 2.

T: All the numbers in the R-image of W are inside this string (trace the string for W on the board), so we can draw the string for R(W) inside the string for W.

Draw a red string inside the one on the board and label it "multiples of 2".



NOTE: Technically, the red string is for the nonnegative multiples of 2. If necessary, point out to the class that with the red string within the string for W, we understand it is for whole number multiples of 2. This note will apply also to other strings within the string for W.

T: Suppose we look at the R-image of the multiples of 2, i.e., the even numbers (trace the red string). Give me an even number and its R-image.

Let several students answer. The following are some possible responses.

S: The R-image of 8 is 16.

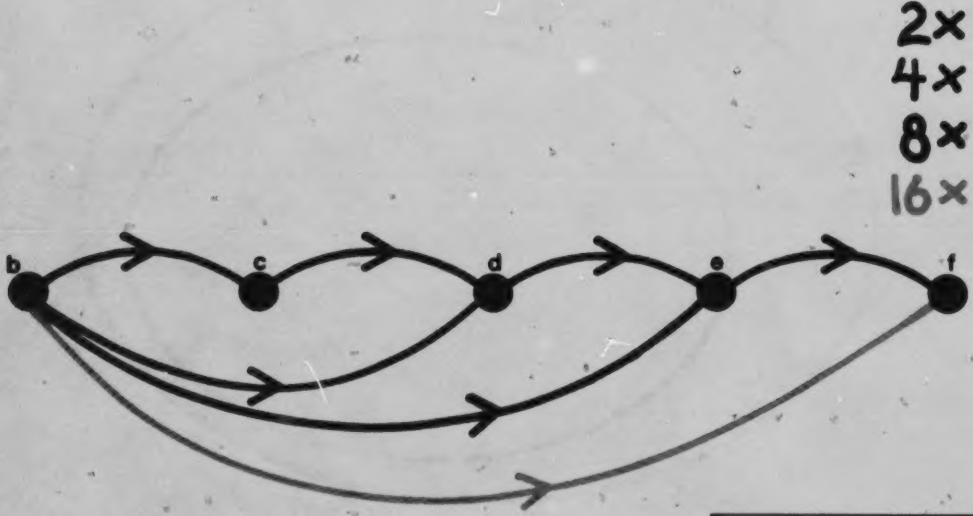
S: The R-image of 2 is 4.

S: The R-image of 10 is 20.

T: What can we say about the numbers in the R-image of the multiples of 2 (trace the red string)?

S: They are multiples of 4 (whole numbers which are multiples of 4).

If your students need help with this question, refer to the arrow picture of Exercise 1.



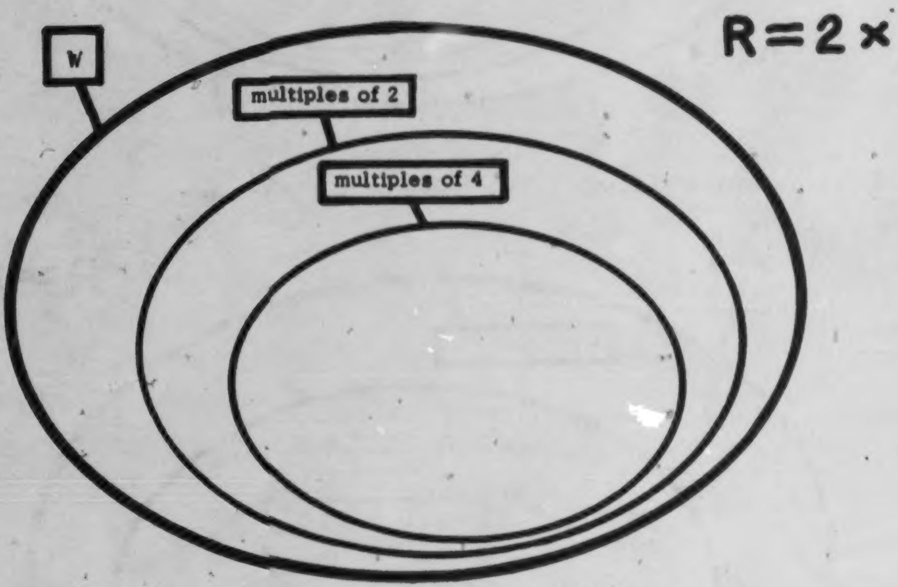
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: If this dot (point to b) is for a whole number, this dot (point to c) is for an even number (that is, a multiple of 2) and this dot (point to d) is for 2 x an even number.

Then ask your original question again.

T: All the numbers in the R-image of the even numbers are inside the red string (trace the red string), so we can draw the string for the R-image of the multiples of 2 inside the red string.

Draw a blue string inside the red string. Label this string "multiples of 4".

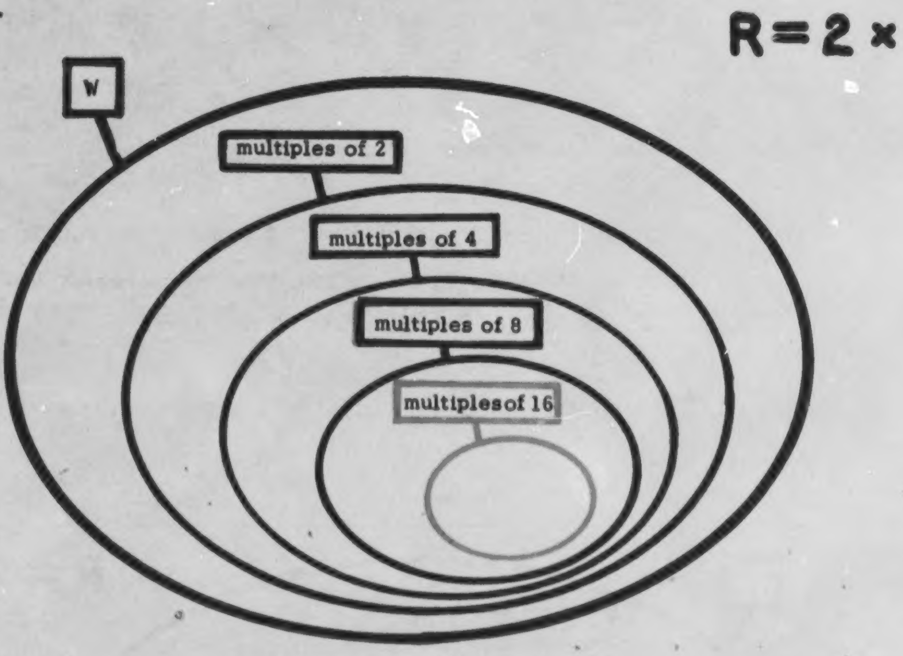


Continue in the same manner for:
- the R-image of the multiples of 4 (multiples of 8)
- the R-image of the multiples of 8 (multiples of 16)

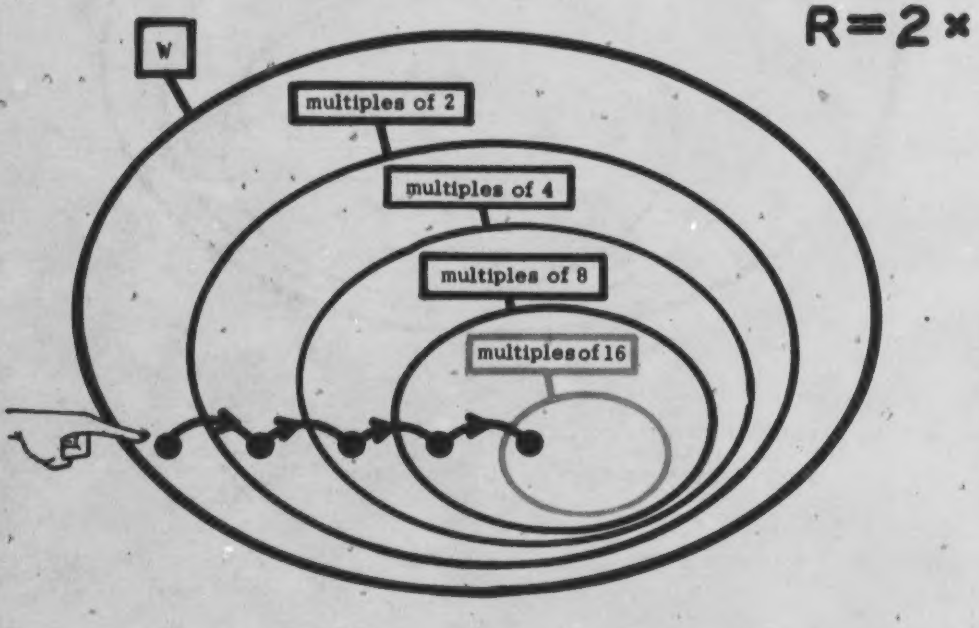
Refer to the arrow picture from Exercise 1 whenever the students need help.

L 11

A complete picture is given below. Note the color of the strings and their labels.


R=2×
W
multiples of 2
multiples of 4
multiples of 8
multiples of 16

Draw these arrows and dots in your picture, and then point to the left most dot.

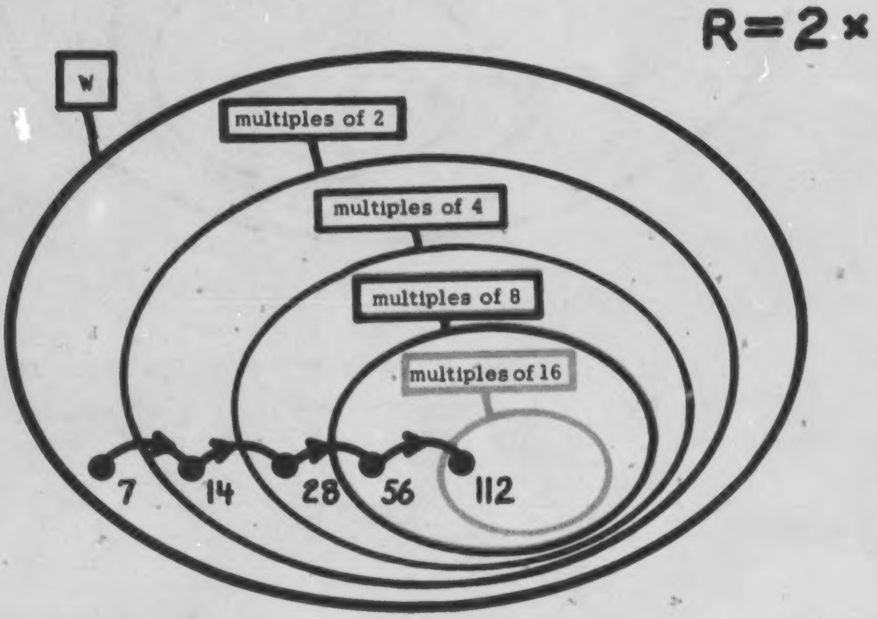

R=2×
W
multiples of 2
multiples of 4
multiples of 8
multiples of 16

T: What kind of numbers could be here?

S: Any odd number.

If necessary, ask " The even numbers are in this string (trace the red string), so what kind of number is here (tap the left most dot)?"

Invite a student to choose an odd number and label the dot. Then let the class proceed to label the other dots in the picture. Repeat this activity several times. An example is given below.

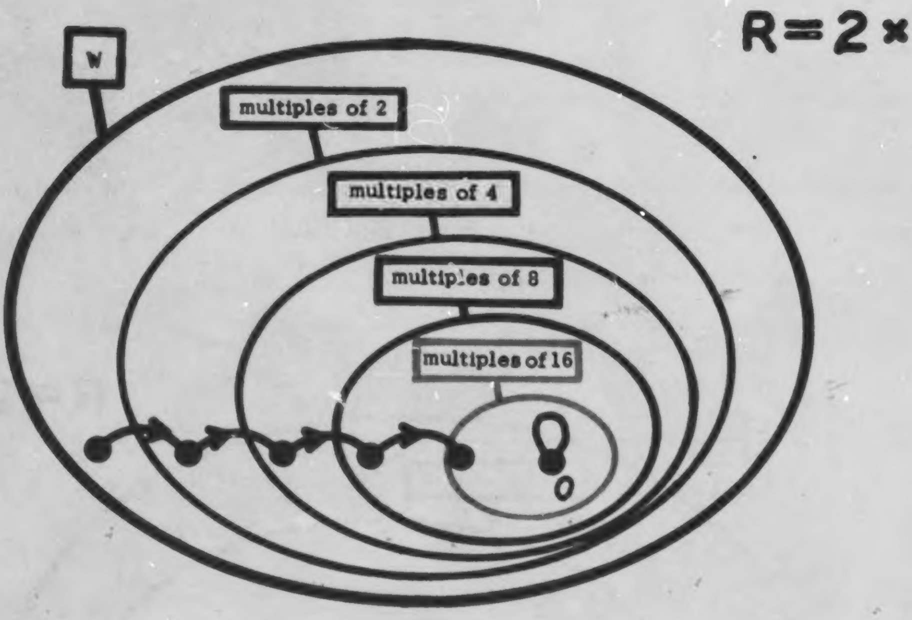

R=2x
W
multiples of 2
multiples of 4
multiples of 8
multiples of 16
7
14
28
56
112

T: If we continued taking images and drawing more strings, is there any number which would be in all the strings?

S: 0, because there is a 2 x loop at 0.

L 11

Draw a dot for 0 and the loop inside the yellow string.


R=2×
W
multiples of 2
multiples of 4
multiples of 8
multiples of 16
0

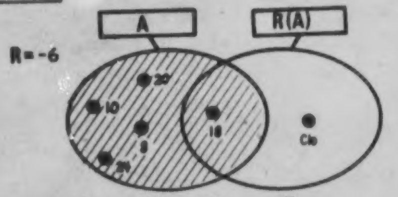
Worksheets L09, L10, and L11 are available for independent work. If you think it would be helpful for most of your students, go over Worksheet L09 collectively.

122

Name____________________ L 09 *

Clo is a secret number.

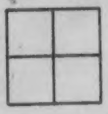
First clue


A
R(A)
R=-6
20
10
8
24
18
Clo

Clo could be _2_, _4_, _12_, or _14_.

Second clue

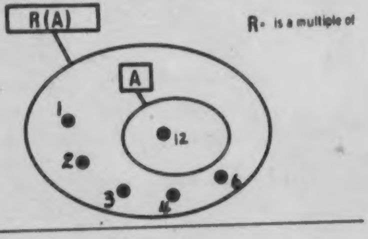
Clo cannot be put on this Minicomputer board with exactly two regular checkers.

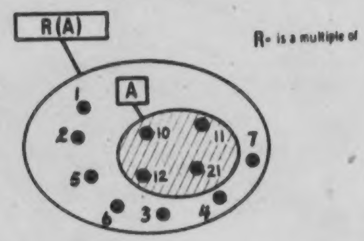


Clo is _14_.

Name____________________ L 10 **

In each problem, label the dots with whole numbers in R(A).

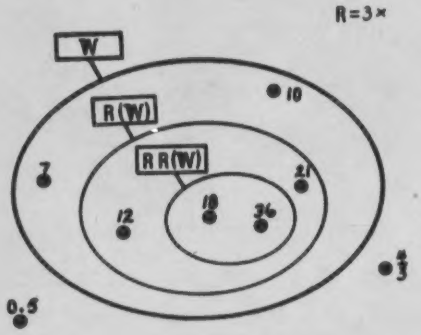

R(A)
R= is a multiple of
A
1
12
2
3
4
6


R(A)
R= is a multiple of
A
1
2
10
11
7
5
12
21
6
3
4

Name____________________ L 11 ***

W is the set of all the whole numbers.
Label four of the dots with these numbers.

12 0.5 7 18


R=3×
W
10
R(W)
RR(W)
7
21
12
18
36
4/3
0.5

Label the remaining four dots with other numbers of your choice.

(Many solutions are possible.)

2.5 Guess My Rule #2

CAPSULE LESSON SUMMARY

Play a Guess My Rule game where the rule is for a function R on the set of points in the plane. Apply this rule to find the R-images of a variety of sets of points. Given an image set, determine some possibilities for a set of points with the given set as its R-image.

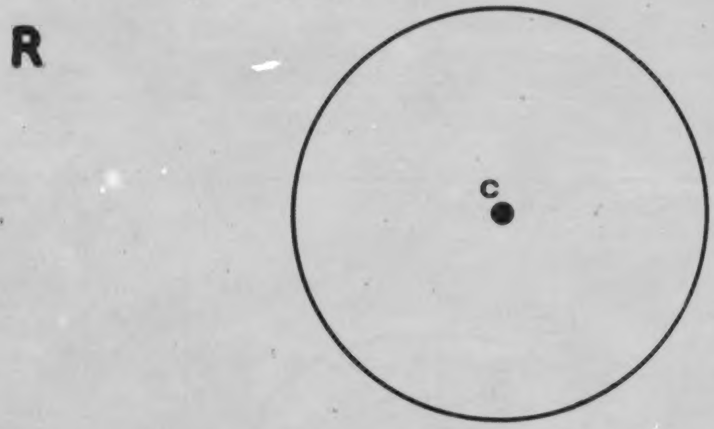
MATERIALS

Teacher: Colored chalk; meter stick or other straightedge
Student: Straightedge or ruler; colored pencils; Worksheets L12, L13, L14, L15, L16

DESCRIPTION OF LESSON

Exercise 1

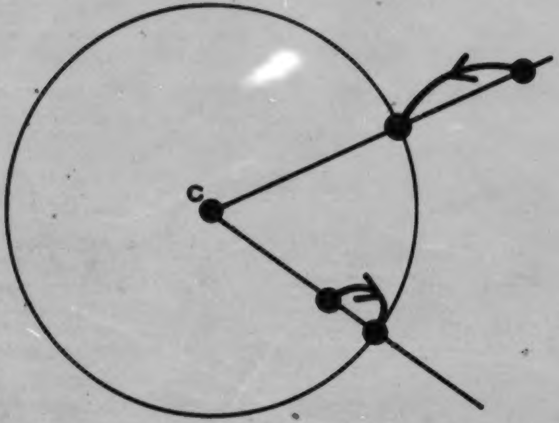
Draw this picture on the board.



T: Let's play a Guess My Rule game. My rule is about points and their images. Here is a special circle with center c. (Point to "R".) My rule is a function R and each point is the starting point of exactly one green arrow. I will give you some clues.

Add this information to your picture. Use a straightedge to construct the rays from the center c of the circle through the points whose images are to be found.

R

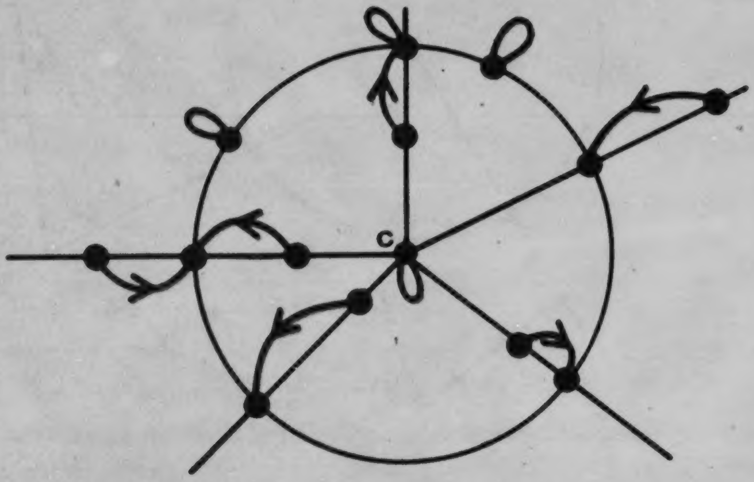


NOTE: The rule to be discovered is:

- the center c is its own image;
- each point of the circle is its own image;
- to find the image of any other point, construct the ray from the center of the circle through the point; the image is that point at which the ray intersects the circle.

One at a time, draw additional points in the picture and ask for the images. After each addition, ask the class to guess how to draw the green arrow starting at the dot. When a student correctly guesses the image of a point, draw the appropriate green arrow or loop. Be sure to include in your discussion the center c of the circle which is special in that it is its own image. Your drawing might look like the following illustration.

R



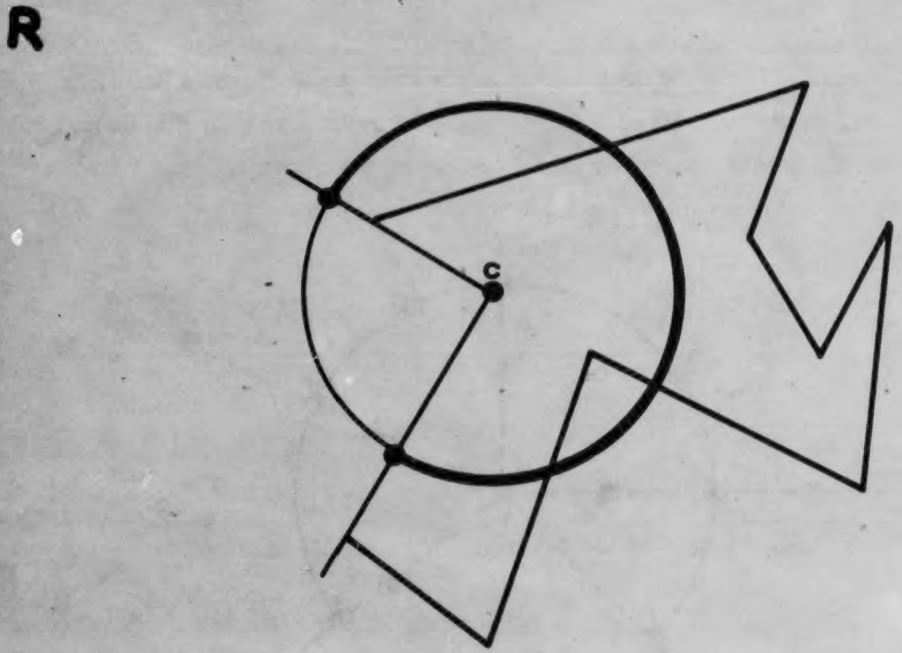
L 12

Exercise 2

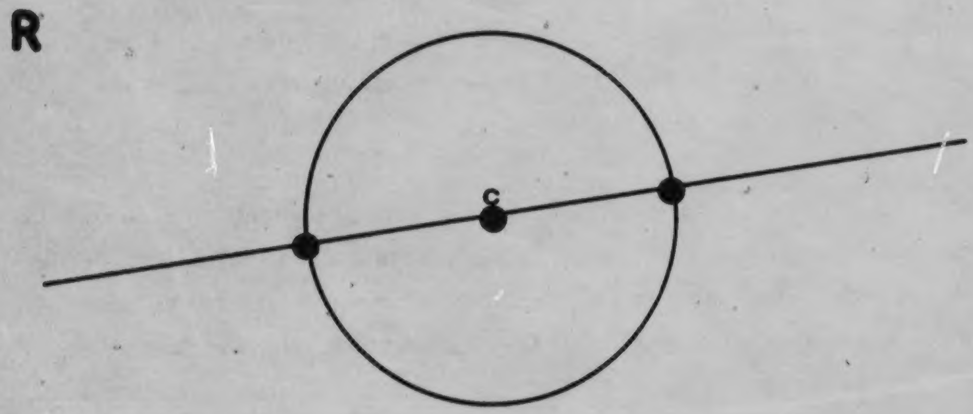
Distribute Worksheet L12.

T: Now that you know the rule of R, use your straightedge and find the R-image of each of the points on this worksheet. Draw the R-image of each point in red and then draw a green arrow from a point to its R-image.

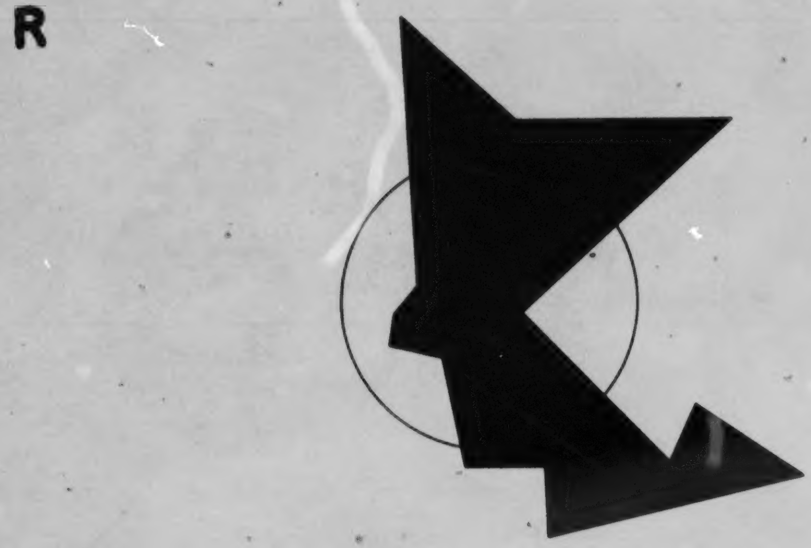
Instruct students who finish Worksheet L12 to continue with Worksheet L13. While students are working, copy the drawing of Worksheet L13 on the board. When most students have begun working on the second problem, initiate a collective discussion to construct the solution on your board drawing. Invite students to find the images of several points that you indicate. Mark the image points in red and continue until the full image of the blue zigzag is known. The next illustration shows the completed picture. Then play Follow My Finger with several students.



Continue this exercise by drawing another circle and a blue line segment through the center of the circle. Play Follow My Finger to determine the R-image of the line segment shown in red here.



Distribute Worksheet L14. Instruct the class to find the R-image of this blue region and color it red. When most students have finished, discuss the solution collectively.



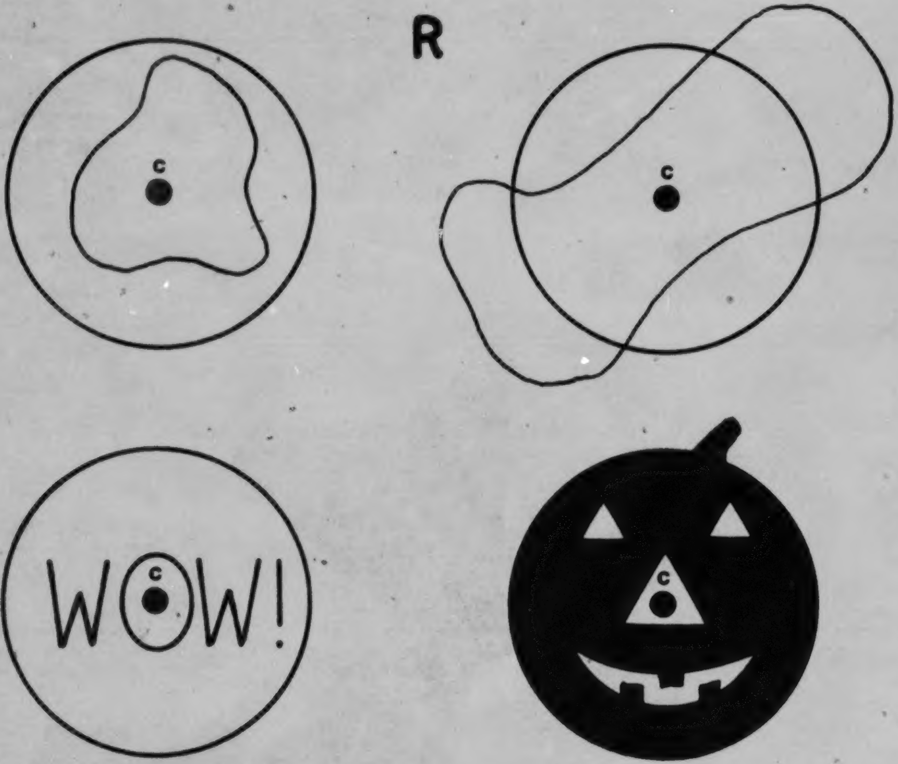
L 12

Exercise 3

Draw a circle on the board and then color it red as shown on Worksheet L15.

T: The special circle is colored red. It is the R-image of many sets. On your worksheet, find a set which has the red circle as its R-image and color it blue.

After a short time for independent work, ask several students to draw their solutions on the board for comparison and discussion. Encourage those who work quickly to find other solutions on their worksheets and to use other colors to show them. Four of the many possible solutions are shown here.

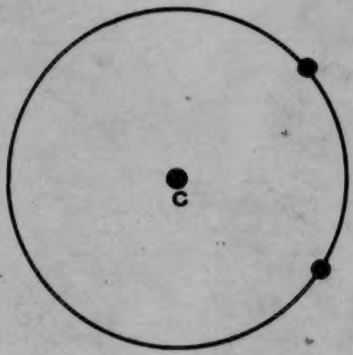


In the discussion, ask the class to determine the largest set whose R-image is the red circle. The discussion of the class' variety of solutions should direct the students to the largest solution; every point of the plane is blue except the center c of the circle. This exclusion is necessary because c is the only point whose image is c. Since c is not red, it follows that c should not be blue.

Exercise 4

Draw the following picture on the board and distribute Worksheet L16.

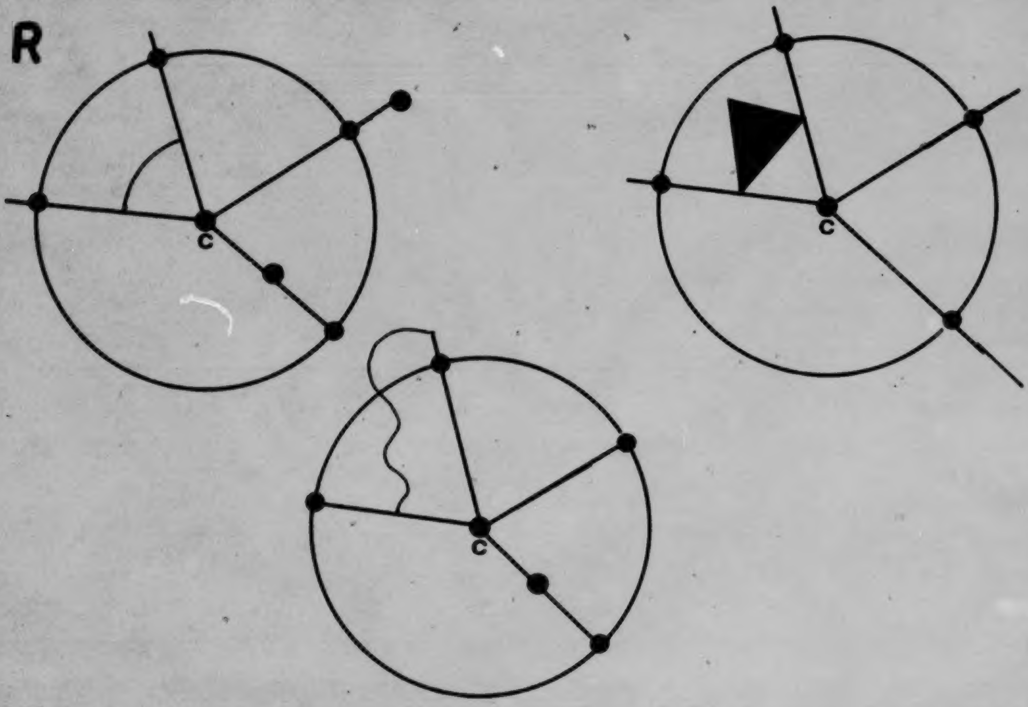
R



T: This set of red points is the R-image of many sets. On your worksheet, find sets whose images are all the red points. Mark each of them with a different color.

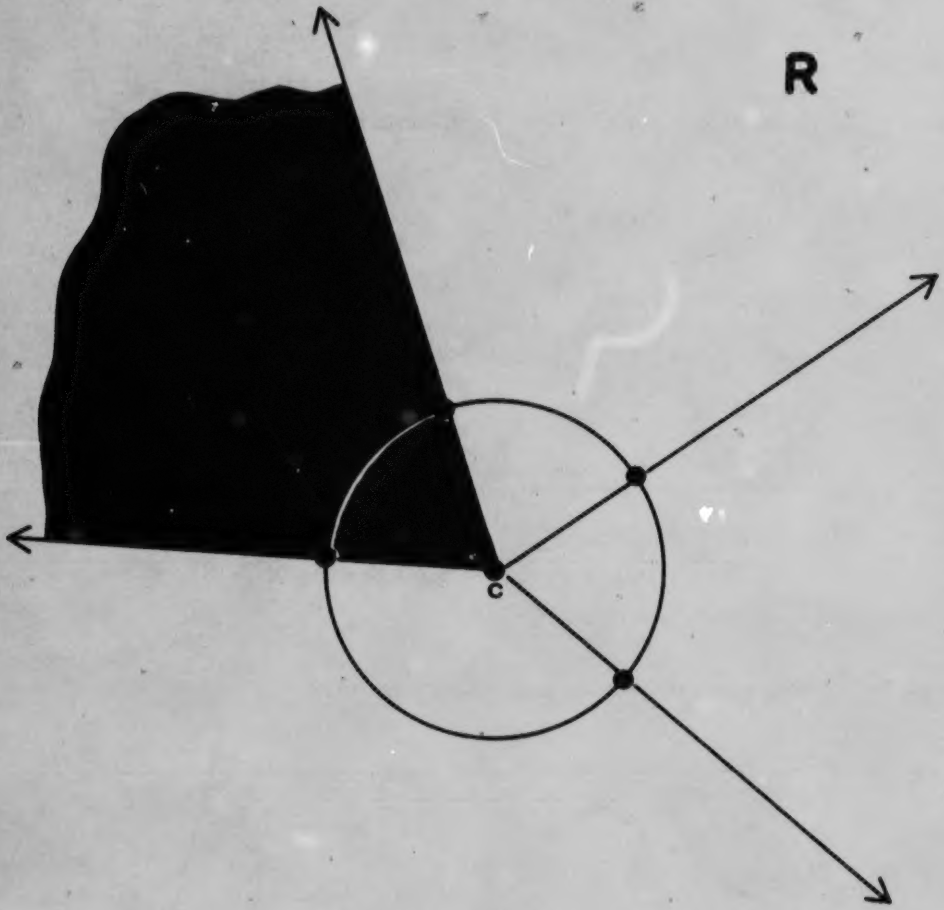
Several of the many possible solutions are shown here.

R



L 12

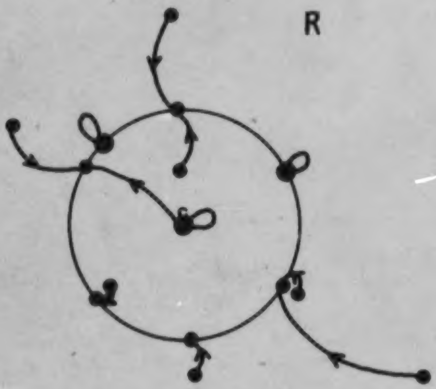
If time remains, ask students to draw some solutions on the board and lead a discussion to find the largest solution, shown below.



130

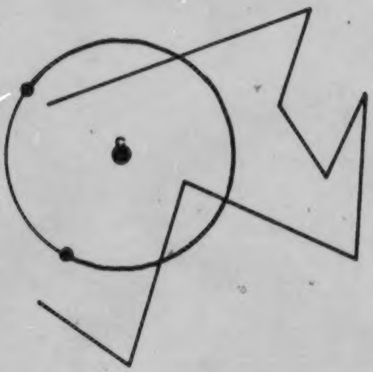
Name ____________ L 12 *

Find the R-image of each point. Draw a green arrow from each point to its R-image. Color the image points red.



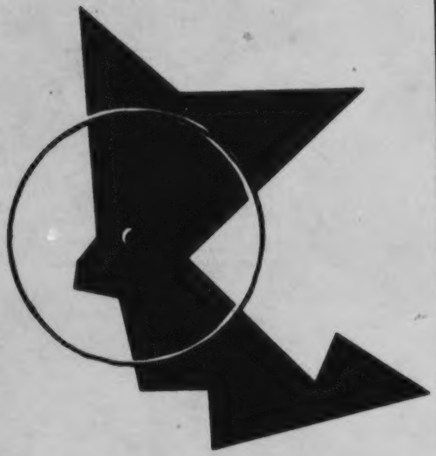
Name ____________ L 13 **

Find the R-image of the blue zigzag. Color the image points red.



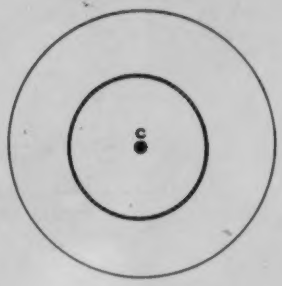
Name ____________ L 14 **

Find the R-image of the blue region. Color the image points red.



Name ____________ L 15 ***

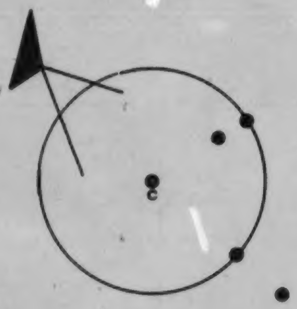
Find a set of points which has the red circle as its R-image. Color it blue.



(Many solutions are possible.)

Name ____________________ L 16 ***

Find a set of points which has the red set as its R-image. Color it blue.



(Many solutions are possible.)

3

DISTANCE

Traditionally, the only concept of distance in the elementary school curriculum is the well-known geometric notion of Euclidean distance - distance "as the crow flies". However, there are many other distances of interest in mathematics and, in fact, the general notion of distance as a special kind of function is an important mathematical concept.

In the CSMP curriculum, students begin thinking about distance (even in Kindergarten) in the simplified settings of taxi-geometry where distance is as a taxi-cab travels and is measured in terms of whole numbers by counting the number of city blocks traveled. Many of the properties of a distance function are experienced in the taxi-geometry activities of the Upper Primary Grades Program. Another kind of distance, decimal distance, was introduced in Part III of the Intermediate Grades Program. In fact, an idea of distance has been underlying many road building activities in previous parts of the Intermediate Grades Program.

In this chapter, the notion of distance is examined in three seemingly unrelated situations. However, one easily notes the similarity of the definition of a distance function in each situation. Properties of distance are explored in each of these situations and the related distance function provides interesting insight into the underlying relations in each case. Using the results about distance, one is able to solve some problems relating to the situations.

CAPSULE LESSON SUMMARIES

Section 3.1: Calculate the lengths of some roads between points on a given map where the lengths of the paths which make up the roads are indicated. Introduce the notion of distance between points and calculate the distance between every pair of points on the map. Solve a problem using the results of the distance calculations.
(Worksheets L17, L18, L19)

Section 3.2: Review the notion of distance on the map from Section 3.1. Given the distance between two points, find a road which has this length. Explore the triangle inequality for distances. Examine the outline of a weighted cube with all the edges having length 1 and determine distances between various pairs of corners. (Worksheets L17, L19, L20)

Section 3.3: Explore the parental relation in a family tree and define parental distance. Find the parental distance between pairs of people in the family tree, and find pairs of people at a given parental distance from each other. Examine family trees for some of the class members. Determine possible relationships between a student and a relative at a parental distance 2 from the student.

Section 3.4: Find pairs of whole numbers which are related by the prime factor relation: i.e., two whole numbers are related if and only if one number equals a positive prime number times the other. Label the dots in some pictures involving the prime factor relation. Define prime factor distance and determine the prime factor distance between pairs of whole numbers. (Worksheets L21, L22)

Section 3.5: Review the definition of the prime factor relation and label dots in a picture involving the prime factor relation. Build roads using the prime factor relation, and find the prime factor distance between some pairs of whole numbers. Determine all the whole numbers less than 50 which are at a prime factor distance of 2 from 10; that is, find the prime factor "circle" with center 10 and radius 2.

3.1 Map Distance #1

CAPSULE LESSON SUMMARY

Calculate the lengths of some roads between points on a given map where the lengths of the paths which make up the roads are indicated. Introduce the notion of distance between points and calculate the distance between every pair of points on the map. Solve a problem using the results of the distance calculations.

MATERIALS

Teacher: Two copies of Language Poster #2 for IG-IV; colored felt tip markers; Worksheet L19

Student: Worksheets L17, L18, L19; pencil

NOTE: Worksheet L19 is an answer key for Worksheet L18. The answer key will be used to solve a problem in Exercise 2. You might wish to remove Worksheet L19 from the worksheet booklets before the lesson begins and distribute it at the appropriate time during the lesson.

DESCRIPTION OF LESSON

NOTE: In this lesson, a distinction is made between the terms "length" and "distance". Each road between two points has a length which is calculated by adding the lengths of the paths which make up the road. The distance between two points is the length of the shortest road between those two points. Try to maintain this distinction as you teach the lesson.

Exercise 1

Display Language Poster #2 for IG-IV.

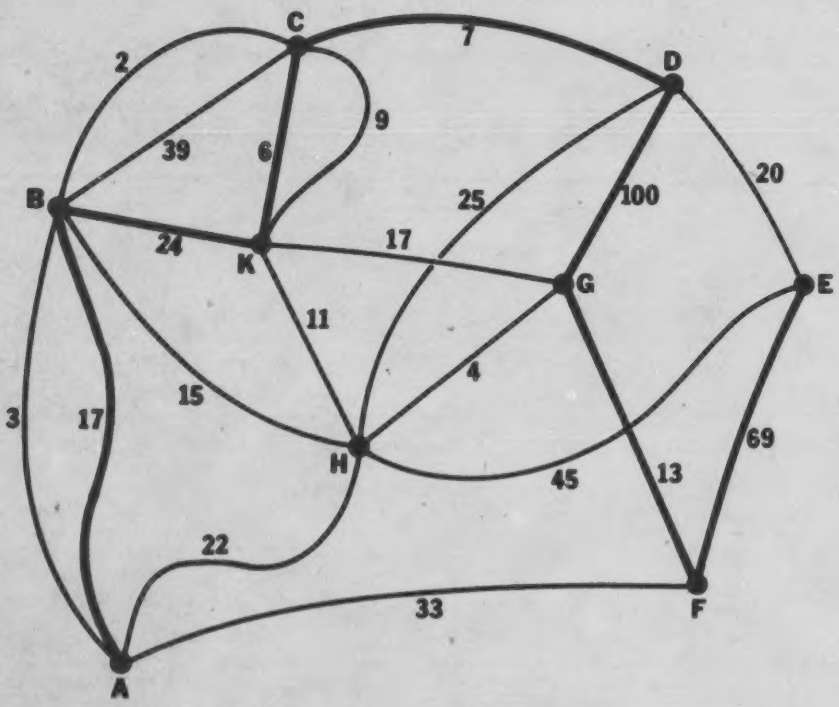
T: This is a map with the length of each path given. You might think that the lengths make it an unusual map. (Pointing to the two direct paths between A and B.) There are two direct paths between A and B; the length of one path is 3 while the length of the other path is 17. We aren't told, but maybe the short path is straight and the long path winds through mountains. Where else on this map do you see two different lengths for direct paths between two points?

S: Between B and C, one path has length 2 and another has length 39.

S: Between C and K, one path has length 6 and the other has length 9.

T: We will pretend to do some traveling on the map and then calculate the length of the road we use. Who would like to draw in red any road starting at A?

Give a student a (red) marker to draw any road starting at A. One possibility is shown in the next illustration.


C
7
D
2
9
39
6
20
B
25
100
24
K
17
G
E
11
4
3
17
15
H
45
13
69
22
33
F
A

T: What is the length of the road drawn in red?

S: 236.

T: How did you calculate that?

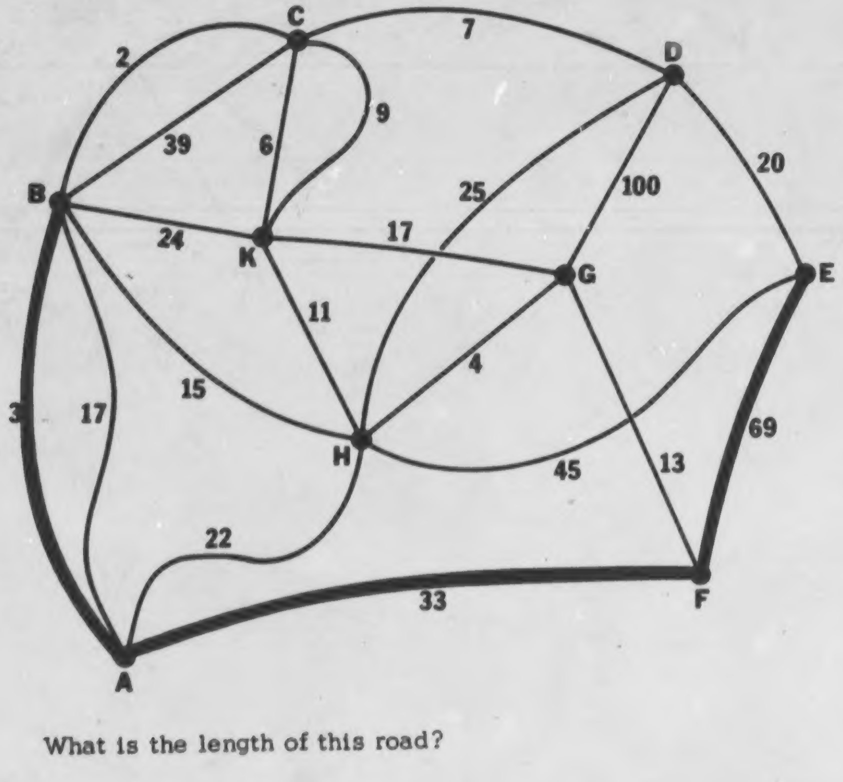
S: I added the lengths of all the paths which make up the road; 17 + 24 + 6 + 7 + 100 + 13 + 69 = 236.

Invite students to draw in different colors one or two more roads starting at A. Ask the class to calculate the length of each of these roads.

Take down the poster and replace it with the second copy of the same poster, Language Poster #2 for IG-IV.

T: Who can slowly trace a road between B and E?

As the student traces a road between B and E, ask the class to calculate its length. A student might suggest the following road.



T: What is the length of this road?

S: 105 (3 + 33 + 69 = 105).

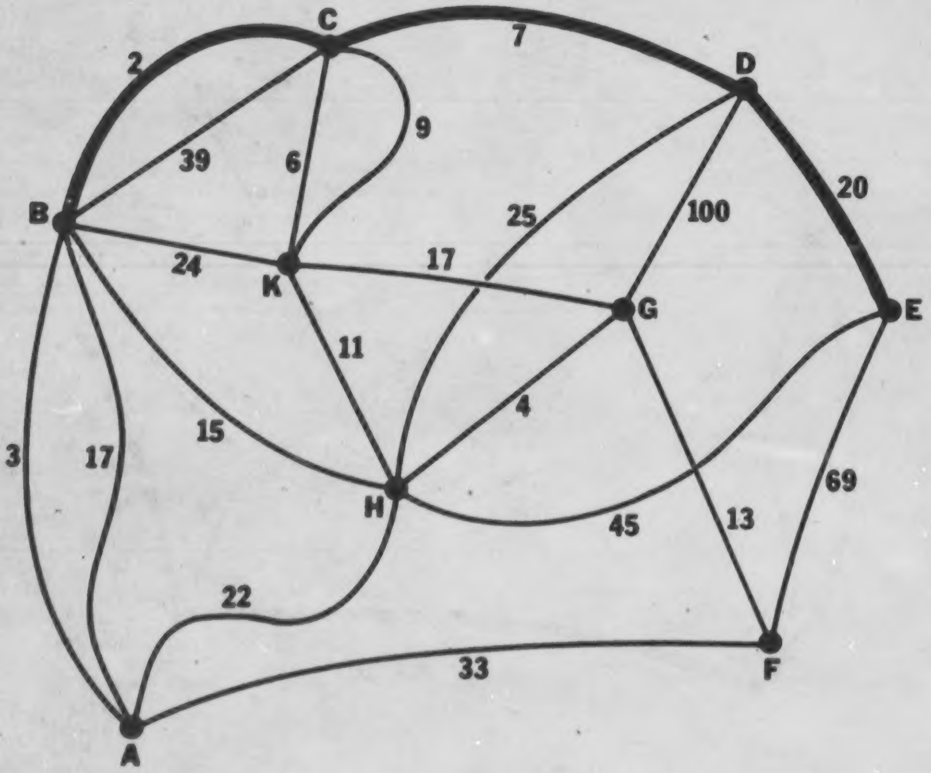
L 4

Record the students' name and the length of the road on the board; for example,

Beth : 105

T: Can anyone trace a shorter road between B and E? What is its length?

Invite several students to trace roads between B and E while the class calculates their lengths. After each student traces a road, write the student's name and the length of the road on the board. Continue this activity until the road of length 29 shown on the next map is suggested. Once the class seems convinced that there are no shorter roads between B and E, confirm that this is the shortest road.



T: This road with length 29 is the shortest road between B and E. Therefore, we say that the distance between B and E is 29.

Write this definition on the board.

> The distance between two points on our map is the length of the shortest road between those two points.

T: Let's find the distance between two more points. Which two points would you like to use?

Let the class find distances between several other pairs of points they suggest in the same manner as they did for B and E. Use the table on Worksheet L19 (which the students receive later) as an answer key. Be sure to include the possibility that both points in a pair can be the same and that the distance from a point to itself is 0.

When you feel the students can continue on their own, distribute Worksheets L17 and L18.

T: Worksheet L17 is a copy of the map. Fill in the table on Worksheet L18 by finding the distance between each pair of points. The distance between B and E, 29, is already recorded for you. You should first record the distances that we just calculated together.

Note that the table is divided into two parts by a heavy black line. To save time, divide the class in half and assign each part of the table to the students in one-half of the class for individual work. Check the students' work, using Worksheet L19 as an answer key.

Exercise 2

When many students have finished their part of the table, distribute Worksheet L19 and ask the students to check their answers.

T: This completed table indicates the distance between every pair of points on the map. We will use this table to solve a problem about Snoopy, Woodstock, and a ferocious cat. Snoopy is very cautious and wants to be as far from the dangerous cat as possible. Woodstock wants to be as close to Snoopy as possible, but they cannot be at the same point. Use the map and table to determine where Snoopy, Woodstock, and the cat should be placed to satisfy both of these conditions.

Make sure that the students understand the problem. Let the class work individually for a while. After many students have found the correct locations (Snoopy at F, Woodstock at G, and the cat at E), solve the problem collectively.

T: What should we do first?

S: Make Snoopy and the cat as far apart as possible. Look for the largest distance in the table.

T: What is the largest distance in the table?

S: 61, the distance between E and F.

T: What can we conclude about Snoopy and the cat?

S: Snoopy is at E and the cat is at F, or the cat is at E and Snoopy is at F.

T: We don't know yet whether Snoopy should be at E or F. The second clue tells us about Woodstock. How does this help us find where Snoopy should be?

S: Woodstock must be as close to Snoopy as possible.

T: How do we use this information?

S: Snoopy must be at E or F. To place Woodstock, find the point which is the smallest distance from E or F. Look at the two rows (or columns) of the table for E and F and find the smallest number. It is 13, the distance from F to G. Therefore, Woodstock should be at G, Snoopy at F, and the cat at E.

Save the unmarked poster for the next lesson on distance. Also save the students' copies of Worksheets L17 and L19.

Worksheet L19 is an answer key for Worksheet L18.

Name ____________________ L 19

	A	B	C	D	E	F	G	H	K
A	0	3	5	12	32	33	22	18	11
B	3	0	2	9	29	32	19	15	8
C	5	2	0	7	27	34	21	17	6
D	12	9	7	0	20	41	28	24	13
E	32	29	27	20	0	61	48	44	33
F	33	32	34	41	61	0	13	17	28
G	22	19	21	28	48	13	0	4	15
H	18	15	17	24	44	17	4	0	11
K	11	8	6	13	33	28	15	11	0

3.2 Map Distance #2

CAPSULE LESSON SUMMARY

Review the notion of distance on the map from Lesson L3.1. Given the distance between two points, find a road which has this length. Explore the triangle inequality for distances. Examine an outline map of a cube with all the edges having length 1 and determine distances between various pairs of corners.

MATERIALS

Teacher: Language Poster #2 for IG-IV; colored chalk
Student: Worksheets L17, L19, L20; pencil

DESCRIPTION OF LESSON

NOTE: The same distinction between "length" and "distance" that was mentioned at the beginning of Lesson L3.1 applies to this lesson.

Exercise 1

Display a copy of Language Poster #2 for IG-IV.

T: Do you remember the funny map from last week's lesson? What is the definition of the distance between two points on the map?

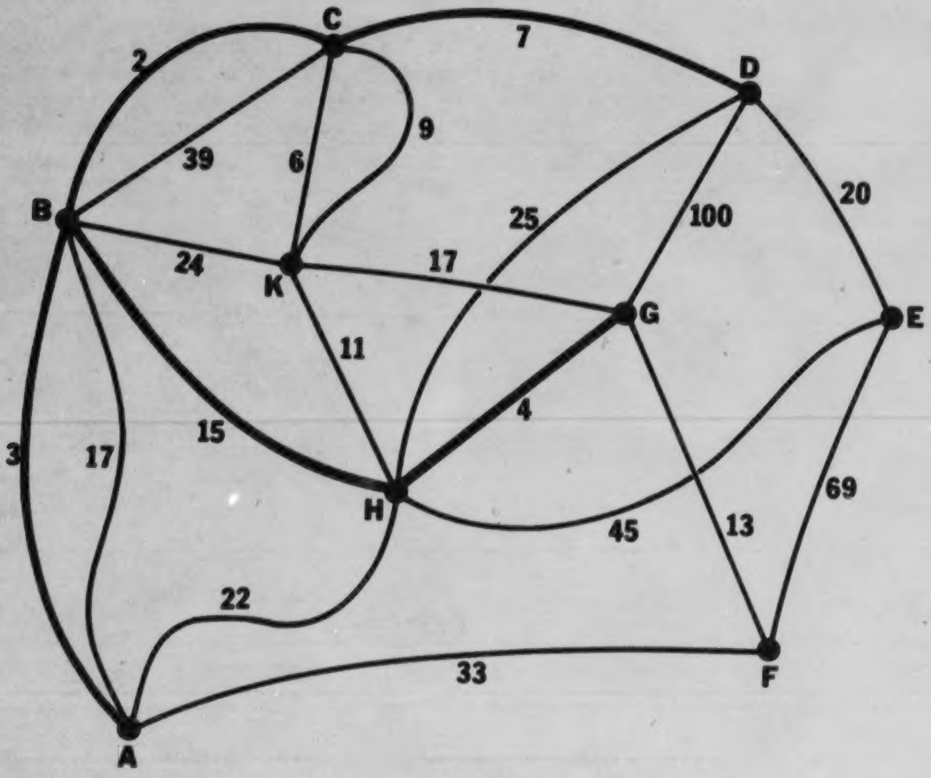
S: The distance is the length of the shortest road between the two points.

T (pointing to "B" and then to "G"): What is the distance between B and G?

Accept suggestions from the students until the class agrees that the distance is 19, the length of the road from B through H to G. See the road shown in red in the next illustration.

T: What is the distance between A and D?

The class should agree that the distance between A and D is 12. The road from A to D of length 12 is illustrated here in blue.


C
7
D
2
9
39
6
B
25
100
20
24
17
K
G
E
11
4
3
17
15
H
45
13
69
22
33
F
A

Distribute Worksheets L17 and L19 which were collected from the students at the end of Lesson L3.1.

T: The table on Worksheet L19 shows the distance between each pair of points on the map. Use the table to find the distance between F and K.

S: 28.

T: How did you find the answer?

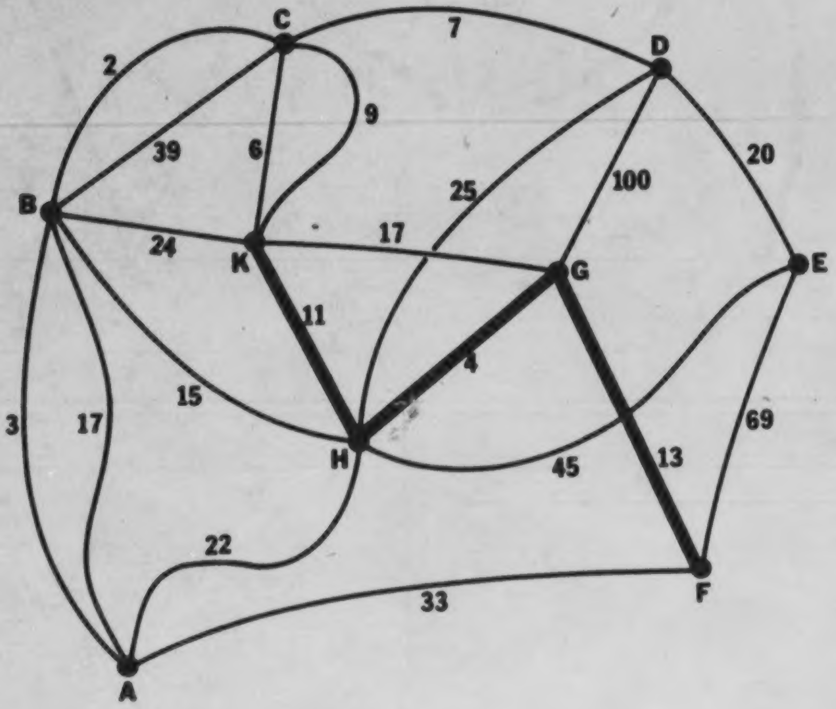
144

S: I looked along the row for F and down the column for K and found the number 28.

S: I used the row for K and the column for F. I got 28 also.

T: Who can show us a road between F and K of length 28?

The correct road is shown in this illustration.


C
7
D
2
9
39
6
20
100
25
B
17
24
K
G
E
11
4
15
3
17
69
H
45
13
22
33
F
A

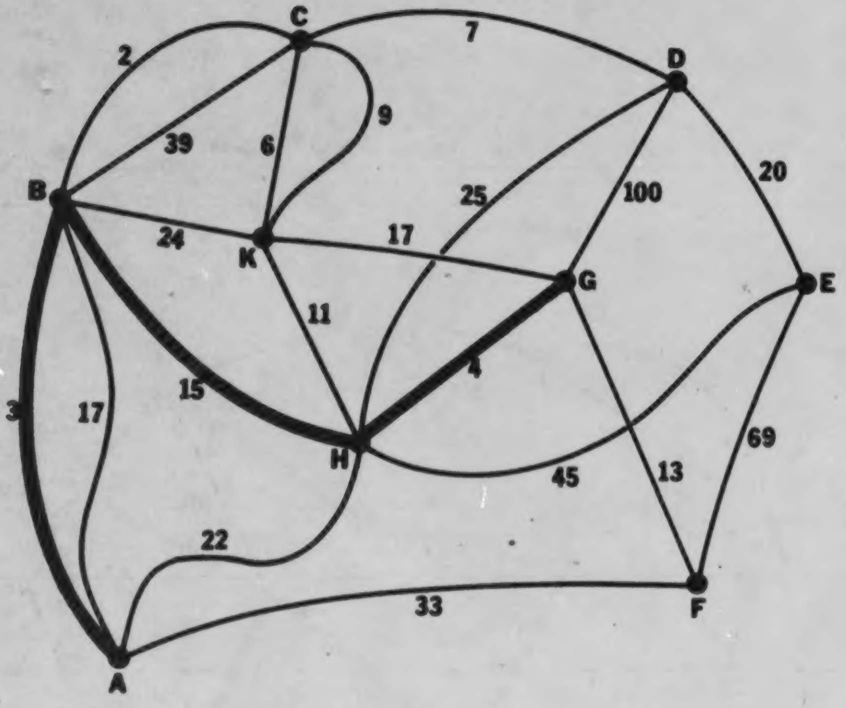
Present the following problem in a similar manner.

T: Use the table to find the distance between A and G.

S: 22.

T: Who can show us a road of length 22 between A and G?

L 5


C
7
D
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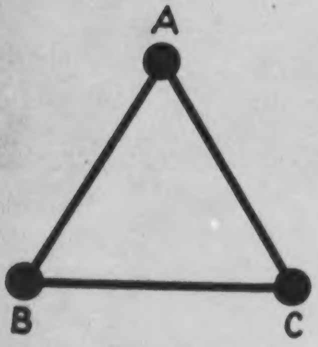
9
39
6
20
B
25
100
24
17
K
G
E
11
4
15
3
17
H
45
13
69
22
33
F
A

Exercise 2

Draw this picture and table on the board.



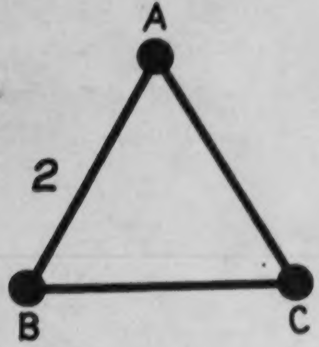
	smallest distance	+	middle distance	< = >	largest distance
A,B,C					

146

T: Use the table of distances on Worksheet L19 to find the distance between A and B.

S: 2.

In the picture, write "2" near the cord between A and B.



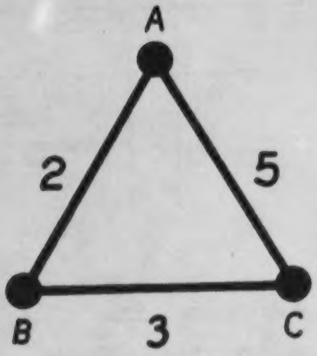
T: What is the distance between A and C?

S: 5.

T: What is the distance between B and C?

S: 3.

Add this information to the picture and table on the board.



	smallest distance	+	middle distance	< = >	largest distance
A,B,C	2	+	3	=	5

T: I put these three distances in the table in the order of increasing size. Which symbol should I put in the fourth column: "<" (smaller than), "=" (equals), or ">" (larger than)?

S: "=", because 2 + 3 = 5.

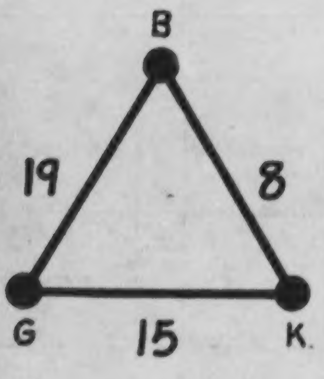
Write "=" in the appropriate space in the table.

T: For A, B, and C, the sum of the two shorter distances equals the longer distance. I wonder if this would be true for any three points on the map.

Erase the labels from the cord picture to prepare for a new labeling.

T: Let's check it with three more points. Which three points do you suggest we use?

Repeat this exercise with the three points suggested by your class. For example, the following picture would result if the points B, G, and K were chosen.



	smallest distance	+	middle distance	< = >	largest distance
A,B,C	2	+	3	=	5
B,G,K	8	+	15	>	19

Depending on the three points used in your class, the sum of the two smaller distances will be either equal to or larger than the largest distance. Record the result in your table.

Distribute Worksheet L20 and read the following directions.

T: Continue this activity on your worksheet. The first line of the table is done for you using points A, B, and C. For each line of the table select any three points you wish and fill in the table as we just did together. Try to find at least one more example using the equal sign and some examples not using the equal sign.

Let the students work individually. For your reference, a complete answer key with all possible combinations of three points is included at the end of this lesson. Continue the collective lesson after most students have done several examples.

Erase the board and draw this table.

=	>	<
A,B,C 2 + 3 = 5		

T: Who found another example of three points where the distances combine to give a true number story using "="?

Record the examples of several students in the "=" column of the table on the board.

L 5

=	>	<
A,B,C $2 + 3 = 5$		
D,E,H $20 + 24 = 44$		
C,E,F $27 + 34 = 61$		
F,G,H $4 + 13 = 17$		
C,H,F $17 + 17 = 34$		

T: Who found an example for which the true number story uses ">"?

List the examples of several students in the ">" column of the table. In the unlikely event that no one has found such an example, suggest one yourself (such as points B, G, and K) and solve it with your students' assistance. Then ask the students to find a couple more examples on their own. The table below shows some examples your class might suggest.

=	>	<
A,B,C $2 + 3 = 5$	B,G,K $8 + 15 > 19$	
D,E,H $20 + 24 = 44$	A,F,H $17 + 18 > 33$	
C,E,F $27 + 34 = 61$	A,D,K $11 + 12 > 13$	
F,G,H $4 + 13 = 17$	A,H,K $11 + 11 > 18$	
C,H,F $17 + 17 = 34$		

T: Who found an example for which the true number story uses "<"?

Do not yet tell the class that this is impossible. If a student suggests, for example, "$19 < 8 + 15$" for B, G, and K, remind the student that on Worksheet L20 this would be written "$8 + 15 > 19$" which uses the ">" sign.

If any student suggests an example using "<", simply accept the suggestion and do the problem collectively. The class will find the student's error in this way.

T: Do you think we could find an example of three points for which the true number story uses "<"?

Encourage the class to discuss this question.

Draw three unlabeled dots on the board.

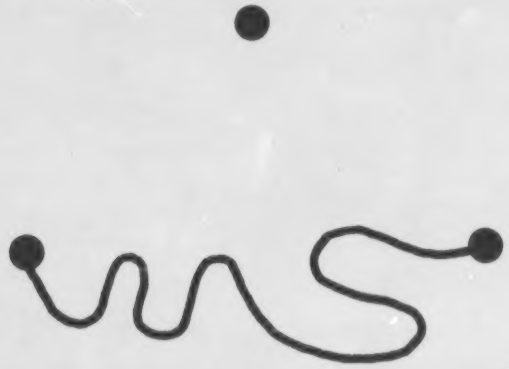
T: These three dots represent any three points on our map. (Point to two of the dots.) How do we find the distance between any two points?

S: Look in the distance table.

T: And how were the numbers in the distance table found?

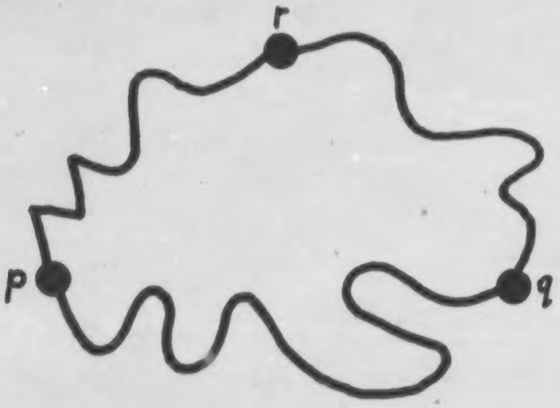
S: Each number is the length of a shortest road between two points.

Draw a green curve between two of the dots.



T: We don't know which is the shortest road between these two points, but let's represent this unknown shortest road by a green curve. It can be crooked because we are using our funny map. The length of this green road is called the distance between these two points.

Similarly, draw these red and blue curves to represent shortest roads between the other pairs of points.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The green curve represents a shortest road between these two points (point to p and q).

This red curve represents a shortest road between these two points (point to p and r).

And the blue curve represents a shortest road between these two points (point to q and r).

Let's suppose that of these three roads, the green road is the longest. (Trace the green curve.) Still we know it is a shortest road between these two points (p and q). Can someone come to the board and trace another road between the same two points?

Invite a student to trace the red curve from p to r and then the blue curve from r to q.

T (pointing to p and q): So we have shown two roads between these two points. We could follow the green road or we could follow the red road and then the blue road.

Write this information on the board.

red distance + blue distance ☐ green distance

T: What sign could I put in the box: =, <, or >?

S: It could be ">", because the green road is a shortest road between those two points. So taking the red road followed by the blue road might be longer.

S: It could be "=", because the red road followed by the blue road might be another shortest road between the two points.

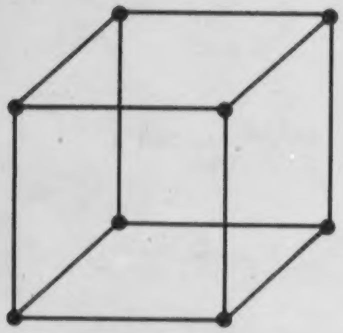
Fill in the box as the students argue for one of the signs.

red distance + blue distance $\geq$ green distance

T (pointing to "$\geq$"): This sign is read "larger than or equal to". This diagram shows us why no one could find an example of three points on our map for which the true number story used "<". No matter which three points are chosen, the sum of the two smaller distances will always be larger than or equal to the largest distance.

Exercise 3

Erase the board and draw this outline picture of a cube.



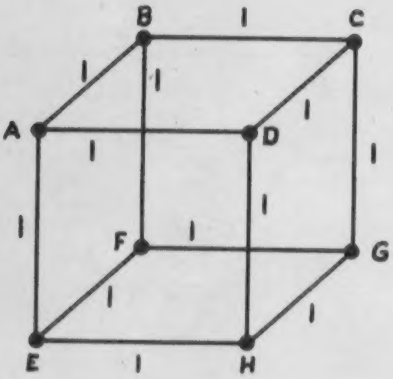
T: This picture shows the outline of a cube. Imagine that pieces of wire have been put together into the shape of a cube. How many corners does a cube have?

S: 8.

T: How many line segments or edges are there?

S: 12.

Label the corners "A" through "H" and label each edge "1".



T: This is a kind of map. The length of each edge is 1. To travel between two points, for example A and G, you must follow the edges. The distance between two points is the length of a shortest road between those two points.

What is the distance between A and C?

S: 2. The shortest roads from A to C are the road from A through B to C and the road from A through D to C. Both of these roads have length 2.

T: What is the distance between A and E?

S: 1.

T: What is the distance between A and G?

S: 3.

T: Which points are at a distance of 1 from A?

S: B, D, and E.

T: Which points are at a distance of 2 from A?

S: C, F, and H.

T: Which points are at a distance of 3 from A?

S: Only G.

155

Name ______________________ L 20 *

Complete the chart. Select any three points you wish for each row. One row is done for you.

three points	smallest distance	+	middle distance	< = >	largest distance
A,B,C	2	+	3	=	5
A,B,D	3	+	9	=	12
A,B,E	3	+	29	=	32
A,B,F	3	+	32	>	33
A,B,G	3	+	19	=	22
A,B,H	3	+	15	=	18
A,B,K	3	+	8	=	11
A,C,D	5	+	7	=	12
A,C,E	5	+	27	=	32
A,C,F	5	+	33	>	34
A,C,G	5	+	21	>	22
A,C,H	5	+	17	>	18
A,C,K	5	+	6	=	11
A,D,E	12	+	20	=	32
A,D,F	12	+	33	>	41
A,D,G	12	+	22	>	28
A,D,H	12	+	18	>	24
A,D,K	11	+	12	>	13
A,E,F	32	+	33	>	61
A,E,G	22	+	32	>	48
A,E,H	18	+	32	>	44
A,E,K	11	+	32	>	33
A,F,G	13	+	22	>	33
A,F,H	17	+	18	>	33
A,F,K	11	+	28	>	33
A,G,H	4	+	18	=	22
A,G,K	11	+	15	>	22
A,H,K	11	+	11	>	18
B,C,D	2	+	7	=	9
B,C,E	2	+	27	=	29
B,C,F	2	+	32	=	34
B,C,G	2	+	19	=	21
B,C,H	2	+	15	=	17
B,C,K	2	+	6	=	8
B,D,E	9	+	20	=	29
B,D,F	9	+	32	=	41
B,D,G	9	+	19	=	28
B,D,H	9	+	15	=	24
B,D,K	8	+	9	>	13
B,E,F	29	+	32	=	61

three points	smallest distance	+	middle distance	< = >	largest distance
B,E,G	19	+	29	=	48
B,E,H	15	+	29	=	44
B,E,K	8	+	29	>	33
B,F,G	13	+	19	=	32
B,F,H	15	+	17	=	32
B,F,K	8	+	28	>	32
B,G,H	4	+	15	=	19
B,G,K	8	+	15	>	19
B,H,K	8	+	11	>	15
C,D,E	7	+	20	=	27
C,D,F	7	+	34	=	41
C,D,G	7	+	21	=	28
C,D,H	7	+	17	=	24
C,D,K	6	+	7	=	13
C,E,F	27	+	34	=	61
C,E,G	21	+	27	=	48
C,E,H	17	+	27	=	48
C,E,K	6	+	27	=	33
C,F,G	13	+	21	=	34
C,F,H	17	+	17	=	34
C,F,K	6	+	28	=	34
C,G,H	4	+	17	=	21
C,G,K	6	+	15	=	21
C,H,K	6	+	11	=	17
D,E,F	20	+	41	=	61
D,E,G	20	+	28	=	48
D,E,H	20	+	24	=	44
D,E,K	13	+	20	=	33
D,F,G	13	+	28	=	41
D,F,H	17	+	24	=	41
D,F,K	13	+	28	=	41
D,G,H	4	+	24	=	28
D,G,K	13	+	15	=	28
D,H,K	11	+	13	=	24
E,F,G	13	+	48	=	61
E,F,H	17	+	44	=	61
E,F,K	28	+	33	=	61
E,G,H	4	+	44	=	48
E,G,K	15	+	33	=	48
E H K	11	+	33	=	44
F,G,H	4	+	13	=	17
F,G,K	13	+	15	=	28
F,H,K	11	+	17	=	28
G,H,K	4	+	11	=	15

3.3 Parental Distance

CAPSULE LESSON SUMMARY

Explore the parental relation in a family tree and define parental distance. Find the parental distance between pairs of people in the family tree, and find pairs of people at a given parental distance from each other. Examine family trees for some of the class members. Determine possible relationships between a student and a relative at a parental distance 2 from the student.

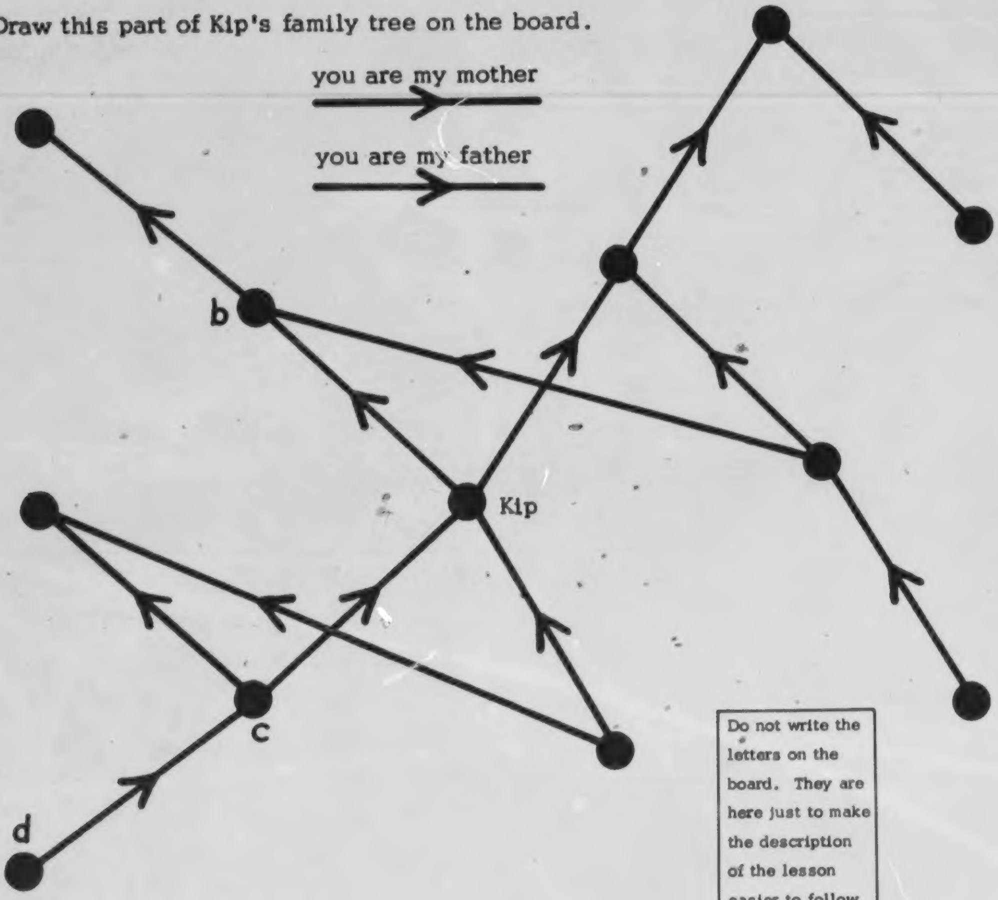
MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Draw this part of Kip's family tree on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

L 8

T: This picture shows some of the people in Kip's family; it is part of his family tree. The red arrows are for "you are my mother" and the blue arrows are for "you are my father".

I put Kip here near the center of the picture. Who is this person (point to b)?

S: Kip's mother because Kip is pointing to her and saying, "You are my mother."

Label b "Kip's mother".

T (pointing to c): Who is this person?

S: Kip's child because that person is pointing to Kip and saying, "You are my father."

T: Can you be more precise than simply "Kip's child"? Do we know more about this person (c)?

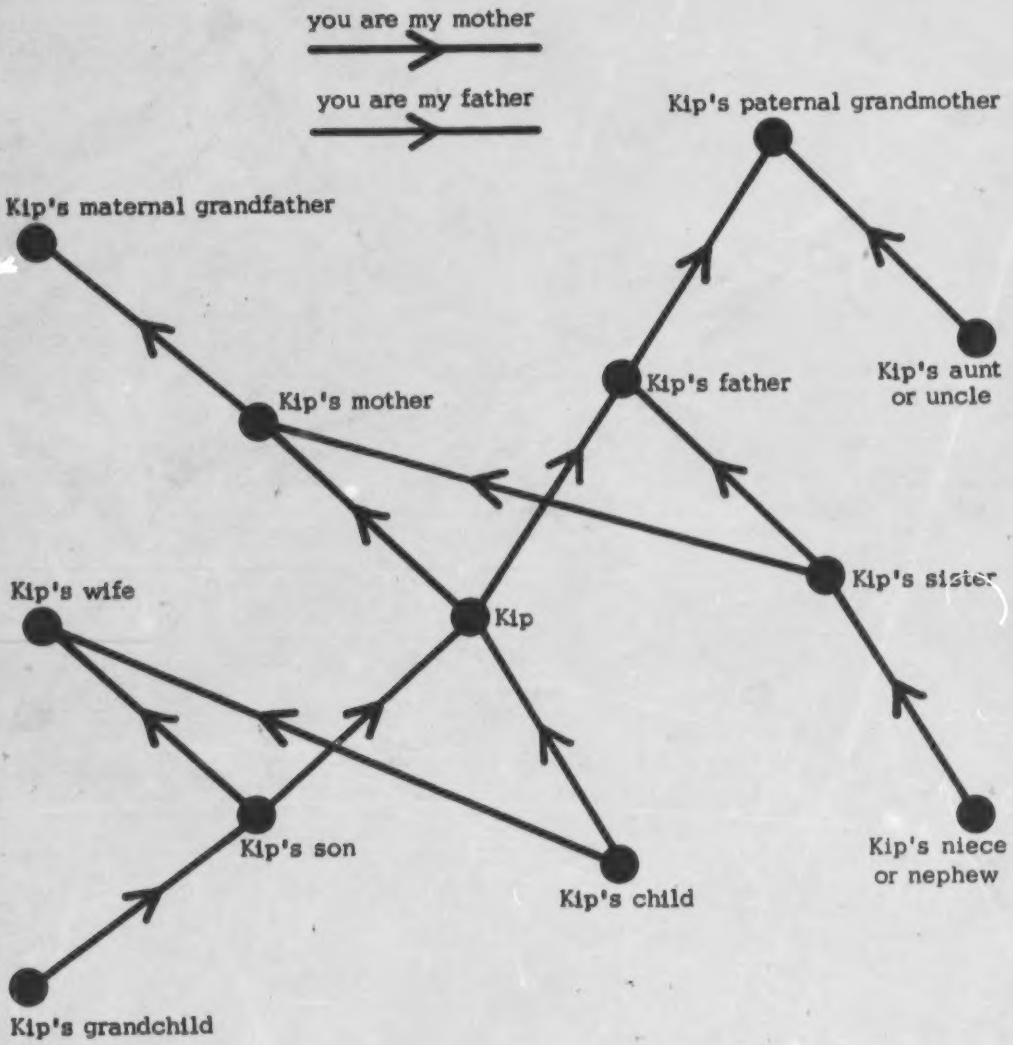
S: Yes, that person is Kip's son. We know he is a son, not a daughter, because another person (d) in the picture is pointing to him and saying, "You are my father."

Label c "Kip's son".

Invite students to label the other dots. The next illustration shows the picture with all the dots labeled. Notice that the sex of a person is known only if the person is the mother or father of someone else in the picture. Therefore, some dots can be labeled specifically, for example, "son", "sister", or "father". However, other dots can only be labeled more generally, for example, "child", "niece or nephew", or "aunt or uncle".

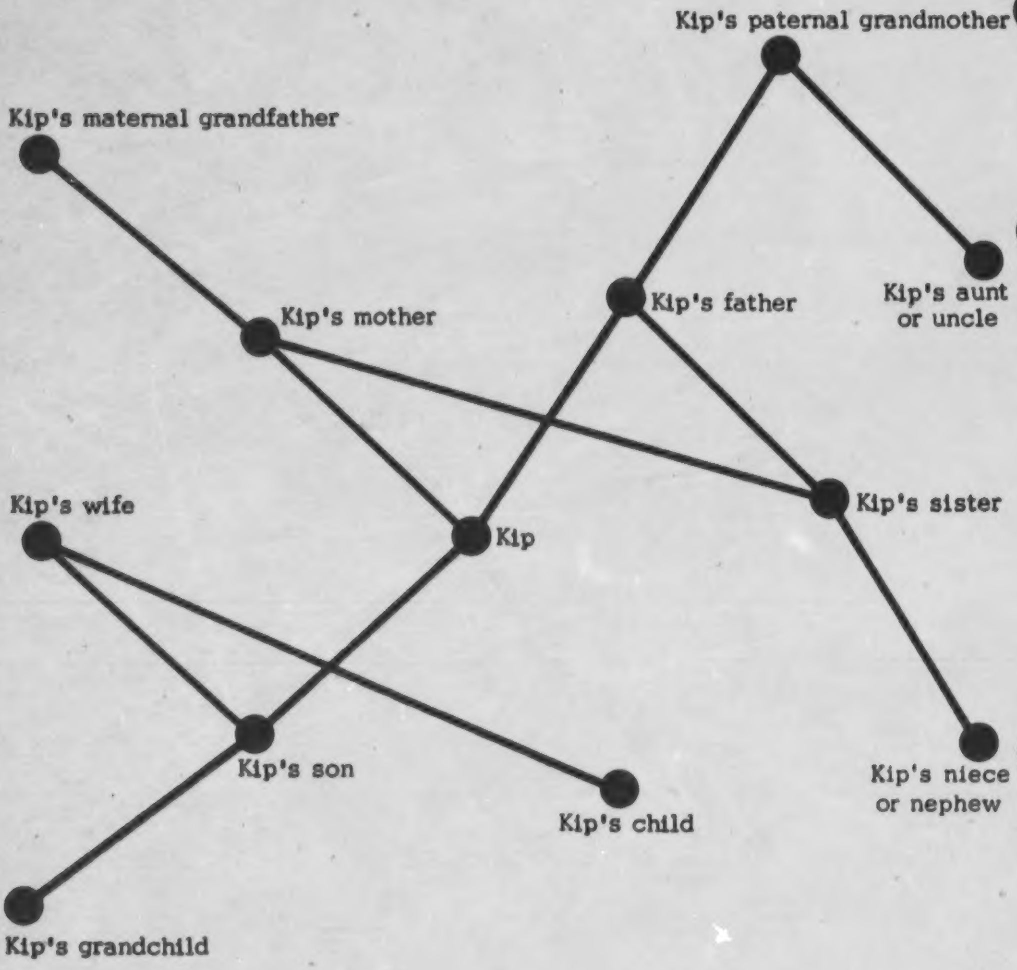
158

L 8


you are my mother
you are my father
Kip's paternal grandmother
Kip's maternal grandfather
Kip's father
Kip's aunt or uncle
Kip's mother
Kip's sister
Kip's wife
Kip
Kip's son
Kip's child
Kip's niece or nephew
Kip's grandchild

Without erasing the dots or labels, replace all of the red and blue arrows by green cords. (See the next illustration.)

L 8


Kip's paternal grandmother
Kip's maternal grandfather
Kip's mother
Kip's father
Kip's aunt or uncle
Kip's wife
Kip
Kip's sister
Kip's son
Kip's child
Kip's niece or nephew
Kip's grandchild

T: What could the green cords be for?

S: "You are my parent or child". Then you can read the green cords in either direction.

Draw a key green cord on the board near the picture.

you are my parent or child

T: This picture is a type of map where each dot is for a person and the cords are for "you are my parent or child".

Who can trace a road following green cords from Kip's wife to Kip's father?

Invite a student to trace any correct road. Then ask another student to trace a road from Kip's niece or nephew to Kip's mother.

Write "Parental Distance" on the board.

T: We say that the parental distance between two people is the number of green cords in the shortest road between those two people. What is the parental distance between Kip and Kip's grandchild?

S: 2, because the shortest road between Kip and Kip's grandchild has two cords.

Similarly, ask students to find the parental distance between each pair of people listed below. To check some of the distances, invite a student to trace a shortest road between the two people.

Kip and Kip's aunt or uncle (parental distance = 3)
Kip's wife and Kip's maternal grandfather (parental distance = 4)
(Your students may notice that there are two distinct roads of length 4 between these two people.)
Kip's son and Kip's sister (parental distance = 3)
(There are two distinct roads of length 3 between these two people.)
Kip's mother and Kip's niece or nephew (parental distance = 2)

T: Can you name a pair of people in this picture who are a parental distance of 3 apart?

L 8

For each pair of people suggested, invite a student to trace a road of length 3 between those two people. Be sure that a shortest road is traced. For example, there is a road of length 3 from Kip's mother to Kip's sister, but the shortest road between them has length 1. Many pairs of people in this picture are at a parental distance of 3, for example: Kip and Kip's aunt or uncle; Kip's maternal grandfather and Kip's niece or nephew; and Kip's wife and Kip's father.

T: What is the largest parental distance between two people in this picture?

Let the class consider this question for a while. For each distance suggested, invite the student to name two people at that distance and to trace a shortest road between them. In this picture, the largest parental distance is 5. It is likely that some students will trace a road longer than 5 between two people, but the class should notice it is not a shortest road between those two people. Remind the class that for parental distance a shortest road must always be used.

T: Yes, the largest parental distance in this picture is 5. Which pairs of people are at a distance of 5?

Trace a road of length 5 for each of the following pairs of people your students suggest.

S: Kip's grandchild and Kip's aunt or uncle.

S: Kip's grandchild and Kip's niece or nephew.

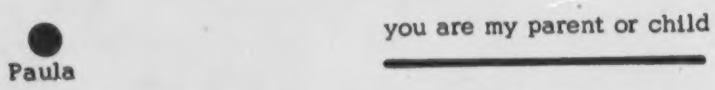
S: Kip's wife and Kip's aunt or uncle.

S: Kip's wife and Kip's niece or nephew.

S: Kip's maternal grandfather and Kip's aunt or uncle.

Exercise 2

Erase the board and write the following information on the board. Label the dot with the name of one of your students, for example, "Paula".


Paula
you are my parent or child

T: Let's use green cords to begin to draw Paula's family tree. Who could be joined to Paula with a green cord?

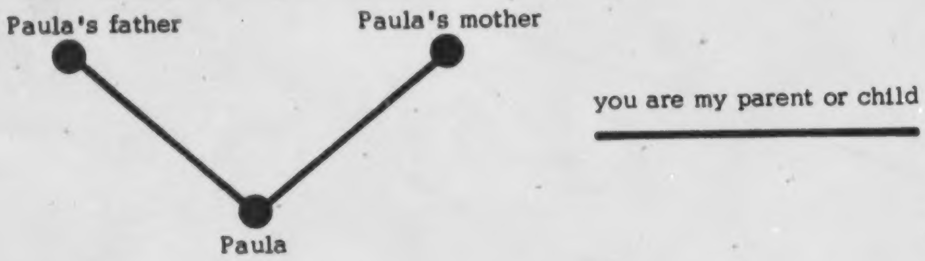
S: Paula's mother and her father.

T: Why?

S: Paula can say, "You are my parent," to both her mother and her father.

S: Paula's mother or father can say, "You are my child," to Paula.

Add this information to your picture.


Paula's father
Paula's mother
you are my parent or child
Paula

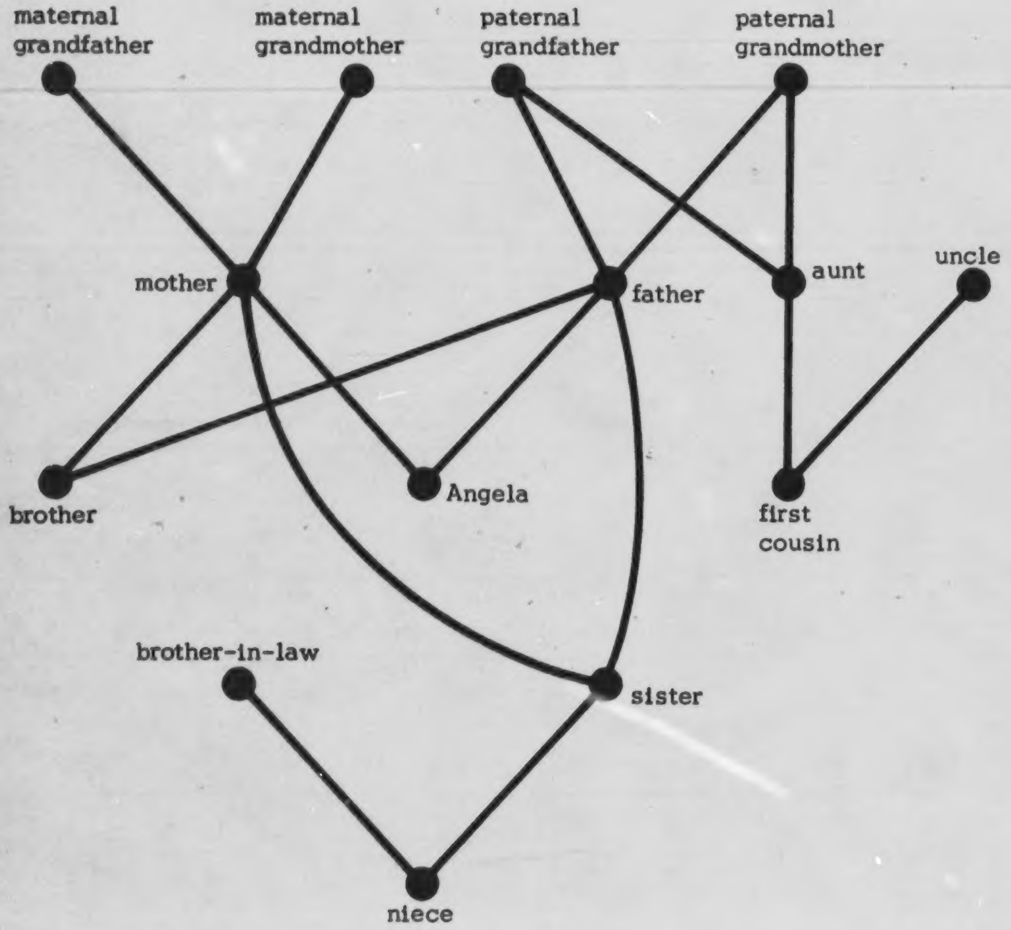
T: We could continue to draw Paula's family tree by adding more relatives and green cords to the picture. But now I would like each of you to draw your own family tree using green cords.

Draw an unlabeled dot on the board.

L8

T: Draw a dot on your paper and write your own name near that dot. Then draw your family tree by adding dots for members of your family. Join them in the correct way with green cords.

Monitor your students' work for a while. Help any students having difficulty by selecting one person in the student's picture and asking, "Can this person say, 'You are my parent,' or 'You are my child,' to any of your relatives?" Then ask the student to add the appropriate dot and green cord. The cord picture below shows most of the types of relatives a student, for example Angela, could include in a family tree. Of course, your students' family trees will vary greatly in the number and types of relatives included. Encourage your students to draw as large a family tree as they can.


maternal grandfather
maternal grandmother
paternal grandfather
paternal grandmother
mother
father
aunt
uncle
brother
Angela
first cousin
brother-in-law
sister
niece

161

Continue the class discussion after most students have drawn much of their family trees.

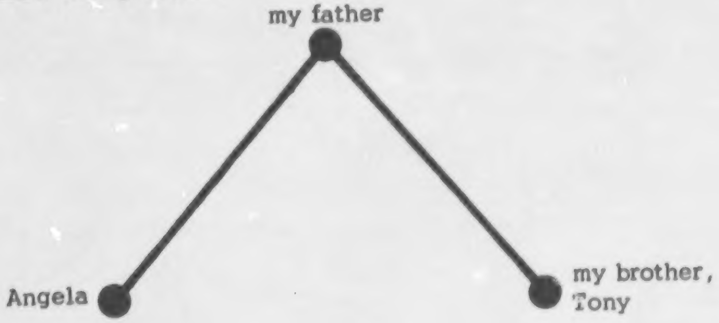
T: Look at your family tree. Can you find someone in your picture at a parental distance of 2 from yourself?

A student might suggest a brother or a sister or a grandparent. Accept suggestions; for example, Angela could suggest her brother, Tony.

S: My brother, Tony, is at a parental distance of 2 from me.

T: Draw a picture on the board to show that your brother is a parental distance 2 from you.

Angela could draw this picture.



T: Angela could add more people to her picture, but this picture is enough to show that her brother is at a parental distance of 2 from her. On your own picture, draw a circle around each person who is at a parental distance of 2 from you.

Give your students some time to do this on their papers.

T: Which people are at a parental distance of 2 from you?

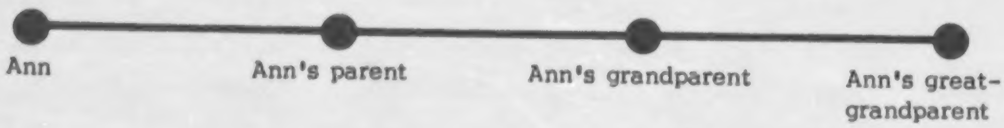
S: My brothers and sisters.

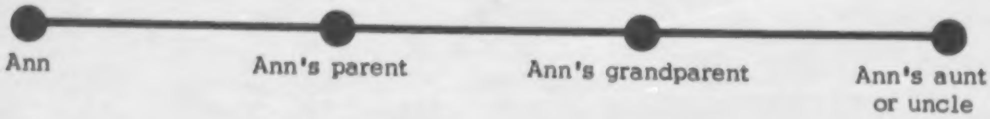
L8

S: All of my grandparents.

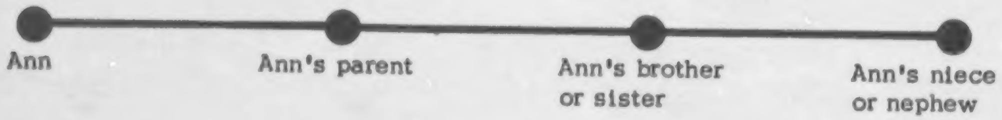
T: Is there anyone in your family tree who is at a parental distance of 3 from you?

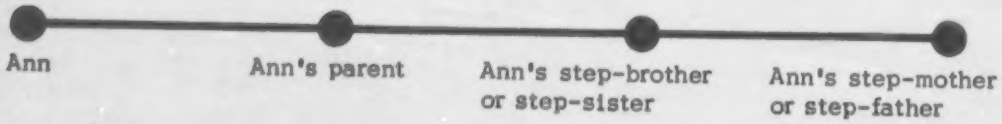
For each person suggested by a student, ask the student to show it on the board by drawing the appropriate part of the family tree. The following pictures show which relatives could be at a parental distance of 3 from a student, Ann. Do not suggest any relationship which your students do not volunteer.


Ann
Ann's parent
Ann's grandparent
Ann's great-grandparent


Ann
Ann's parent
Ann's grandparent
Ann's aunt or uncle

NOTE: Only an aunt (or uncle) who is a parent's sister (or brother) can be included. The spouse of a parent's brother or sister is also an aunt or uncle, but is not at a parental distance of 3 from the given person.


Ann
Ann's parent
Ann's brother or sister
Ann's niece or nephew


Ann
Ann's parent
Ann's step-brother or step-sister
Ann's step-mother or step-father

3.4 Prime Factor Distance #1

CAPSULE LESSON SUMMARY

Find pairs of whole numbers which are related by the prime factor relation; i.e., one number equals a positive prime number times the other. Label the dots in some pictures involving the prime factor relation. Define prime factor distance and determine the prime factor distance between pairs of whole numbers.

MATERIALS

Teacher:	Colored chalk
Student:	Paper; colored pencils; Worksheets L21, L22

DESCRIPTION OF LESSON

Exercise 1

T: Who can give me an example of a positive prime number?

S: 7.

T: Why is 7 a prime number?

S: 7 has only two positive divisors, 1 and 7.

T: Yes, a positive prime number is a positive number with exactly two positive divisors. What are some other positive prime numbers?

S: 2, 3, 5, and 11.

T: What is the smallest positive prime number?

S: 2.

Starting with 2, ask the class to help you list on the board all the positive prime numbers smaller than 30. Since this list will be used throughout the lesson, write it on a part of the board which is out of the way. A list of all positive prime numbers smaller than 30 is given below.

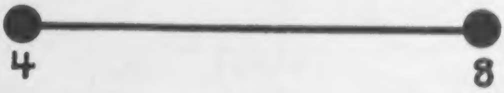
2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Write "PRIME FACTOR RELATION" on the board and draw a red cord.

T: Today, we will use a new relation called the "prime factor relation". The dots are for whole numbers. Two whole numbers are joined by a red cord if and only if one of the numbers equals the other number times a positive prime number. Can you suggest two numbers that we could join with a red cord?

S: 4 and 8.

Label the ends of the red cord "4" and "8".



T: Yes, 8 = 2 × 4 and 2 is a positive prime number. What is another pair of numbers I could join by a red cord?

S: 6 and 18.

T: Why?

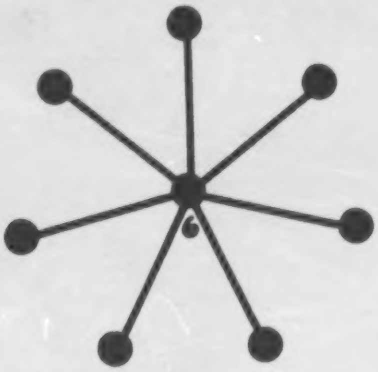
S: 3 × 6 = 18 and 3 is a positive prime number.

Ask for several more pairs of numbers that may be joined by a red cord. Encourage students to find examples using different prime numbers as the multiplier.

T: Can 5 and 20 be joined by a red cord?

S: No, 4 × 5 = 20, but 4 is not a prime number.

Draw this arrow picture on the board.

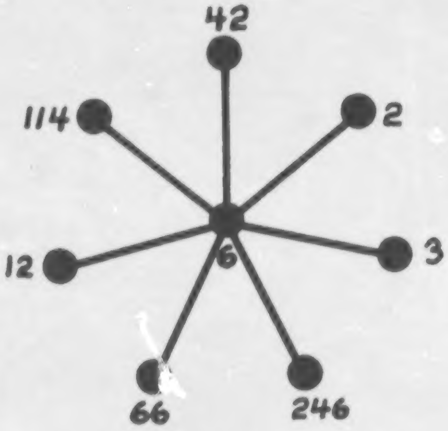


T: Which whole numbers can be joined to 6 with a red cord?

S: 2, because 3 × 2 = 6 and 3 is a positive prime number.

S: 42, because 7 × 6 = 42 and 7 is a positive prime number.

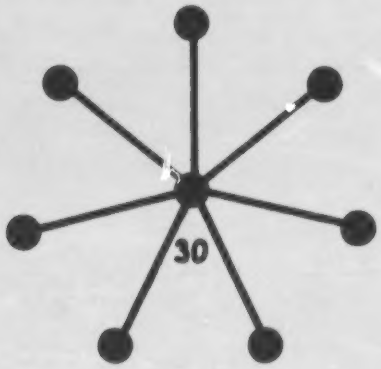
Invite students to label all the dots. Many answers are possible, but encourage the students to suggest numbers both smaller than 6 and larger than 6 that could be joined to 6. A (possible) completed arrow picture is shown below.



169

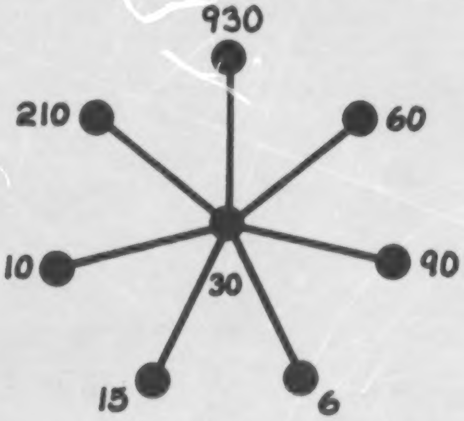
L 14

Erase the numbers from the arrow picture on the board and label the center dot "30".

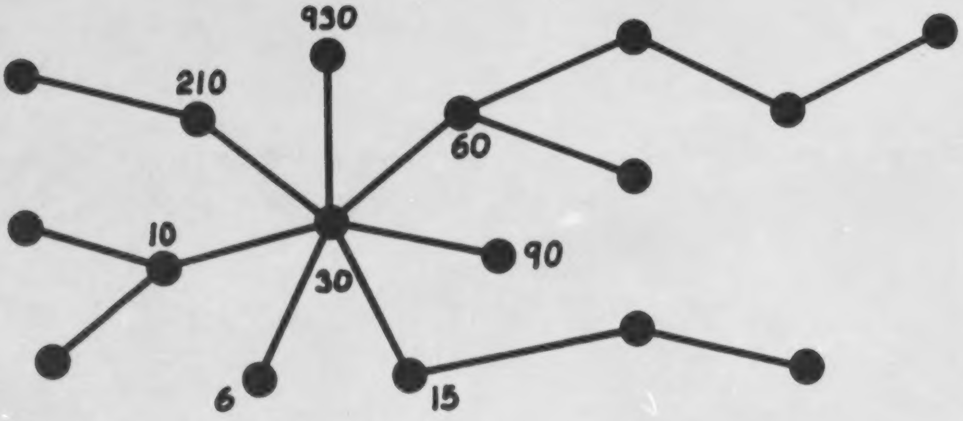


T: Copy this arrow picture on your paper and label the dots with whole numbers. Try to use some numbers smaller than 30 and some numbers larger than 30.

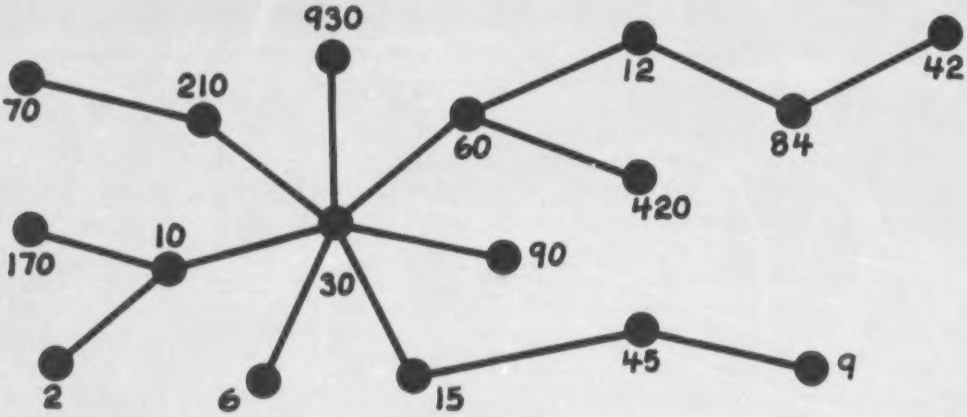
After a while, invite students to label the dots on the board picture. The next illustration includes some of the numbers that could be joined to 30.



Extend your picture in a way appropriate to the numerical ability of your class. For example, the above arrow picture could be extended as in the next illustration.



Invite students to label the dots. Insist that the students follow the convention that each dot in the arrow picture must be for a different number. For some dots, specify that you desire a number that is larger (or smaller) than the number at the other end of the cord. Ask the class to check each answer. A completely labeled arrow picture might look like this.



T: This picture is beginning to look like a map. Do you remember some other maps and distances we studied?

S: There was a funny map with cities and roads. The distance between two points was the length of the shortest road. We solved the problem about Snoopy and Woodstock on that map.

S: There also was the map we got from a family tree. We discussed the parental distance between people.

T: Using the prime factor relation, we have drawn another kind of map. Do you suppose we could define a kind of distance between two numbers on this new map?

S: Count the number of cords in a road between the two numbers.

T: Can we use any road between the two numbers?

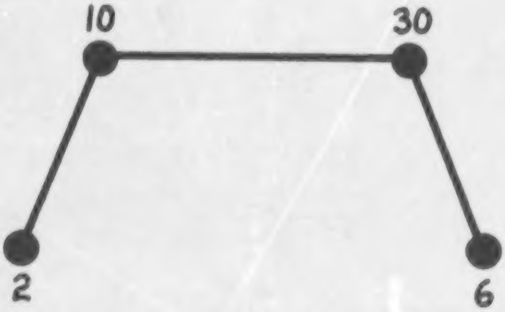
S: No, you must find a shortest road.

T: Yes, we will say that the prime factor distance between two numbers is the number of cords in a shortest road between the two numbers.

Write this information on the board near the large arrow picture.

Prime Factor Distance

pfd (10,6) =



T: This small arrow picture could be part of the larger picture. Let's look at some prime factor distances on this small picture first. What is the prime factor distance between 10 and 6?

S: 2, because there are two red cords in a shortest path from 10 to 6.

T: How do you know that there is no shorter road from 10 to 6?

S: Because 10 and 6 cannot be joined by one red cord.

Complete the equation "p fd (10, 6) = 2".

T: What is the prime factor distance between 2 and 6?

A student might suggest 3 since there is a road from 2 to 6 of length 3 already drawn on the board. If this occurs, ask the following questions.

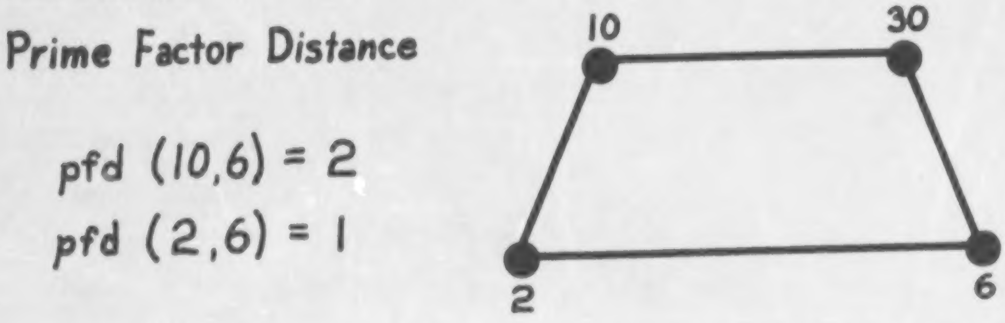
T: There is a road of length 3 between 2 and 6, but is it a shortest possible road? Can someone add a red cord to this picture and show a shorter road between 2 and 6?

S: You can draw a red cord from 2 to 6, since 3 × 2 = 6 and 3 is a positive prime number.

T: What is the prime factor distance between 2 and 6?

S: 1.

Draw a red cord from 2 to 6 and write "p fd (2,6) = 1" near the picture.



T: We must be careful when we try to find a prime factor distance. We must be sure to find the length of a shortest road between two numbers, not just any road.

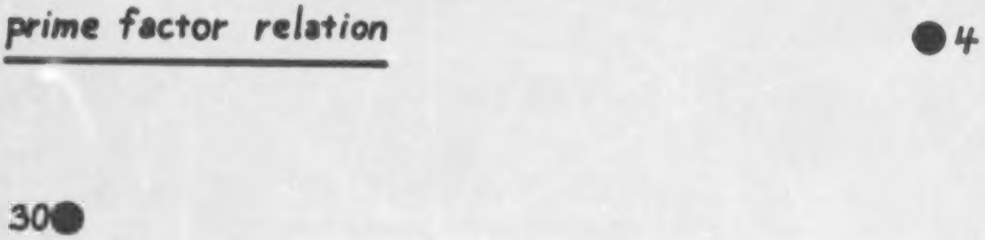
Refer to the large cord picture on the board.

T: In our picture, which numbers are at a prime factor distance of 2 from 30?

The answer depends on the picture drawn by your class. In our picture, 2, 12, 45, 70, 170, and 420 are at a distance of 2 from 30.

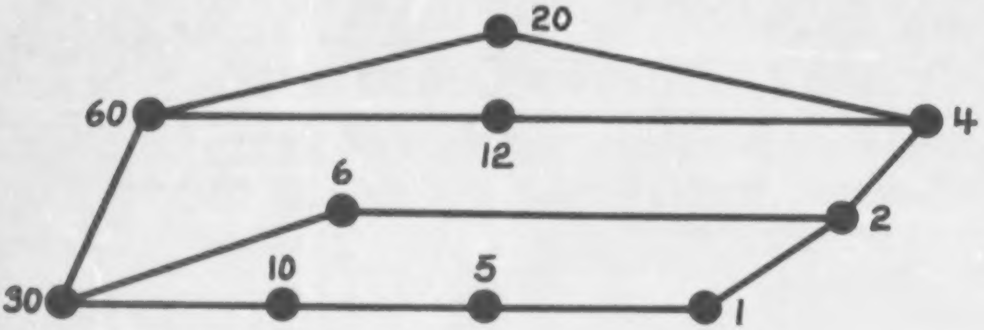
Exercise 2

Erase the board and write this information.



T: Copy this information on your paper and build a road with red cords between 4 and 30. Try to build a shortest road. Build more than one road if you have time.

Let the students work independently for a few minutes. While the others are still working, invite some students to draw their solutions on the board. Attempt to find students who offer differing solutions, not only the shortest roads. For example, your class might draw the following roads.



T: Did anyone find a shorter road between 4 and 30?

S: No.

T: What is the prime factor distance between 4 and 30?

S: 3, because there are three cords in a shortest road between 4 and 30.

If none of the roads between 4 and 30 drawn in your class has length 3, challenge the students independently to find such a road. Then add one of the suggested roads to the picture and conclude that the prime factor distance between 4 and 30 is 3. Write "pfd (4,30) = 3" on the board.

NOTE: If most students were able to find at least one road between 4 and 30, continue with Exercise 3A. But if many had difficulty with that problem, proceed to Exercise 3B.

Exercise 3A

Write this information on the board.

pfd (2,60) =

pfd (4,18) =

pfd (8,9) =

pfd (8,12) =

pfd (15,100) =

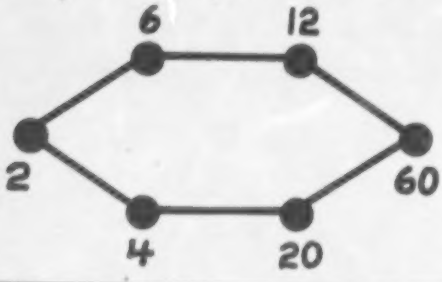
T: We found that the prime factor distance between 4 and 30 is 3. Find these prime factor distances in a similar manner. For each problem, try to build a road between the two numbers and then check if it is a shortest road.

L 14

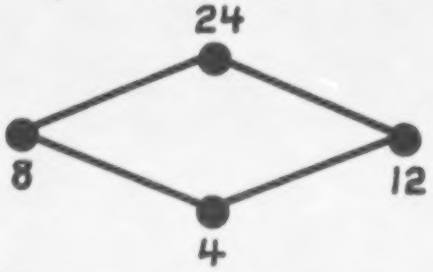
Let your students work on these problems independently. After a while, invite students to draw several roads for each problem on the board. If some students find these problems too difficult, give them Worksheets L21 and L22 to do first.

Near the end of the period, call the class' attention to the roads drawn on the board. Invite any student who has found a shorter road for one of the problems to add that road to the picture. Then ask the class for the prime factor distance between each pair of numbers. The prime factor distance and two of the shortest roads for each problem are presented below.

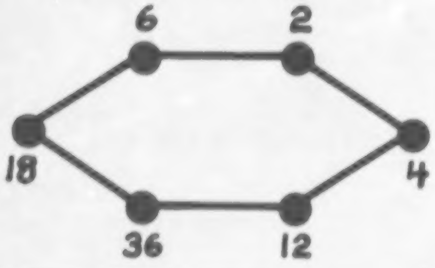
pfd (2,60) = 3



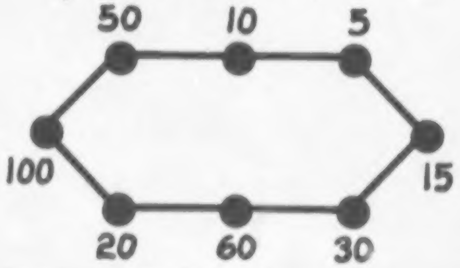
pfd (8,12) = 2



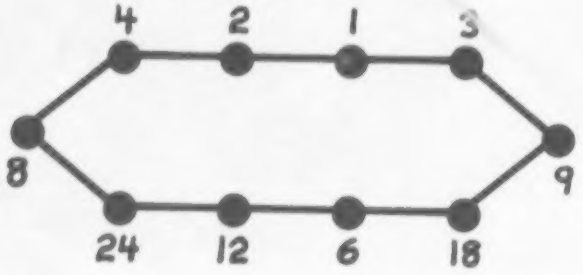
pfd (4,18) = 3



pfd (15,100) = 4



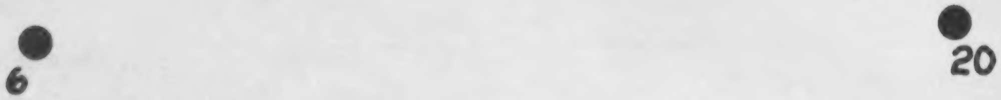
pfd (8,9) = 5



Exercise 3B (This exercise is only for a class that had difficulty in Exercise 2.)

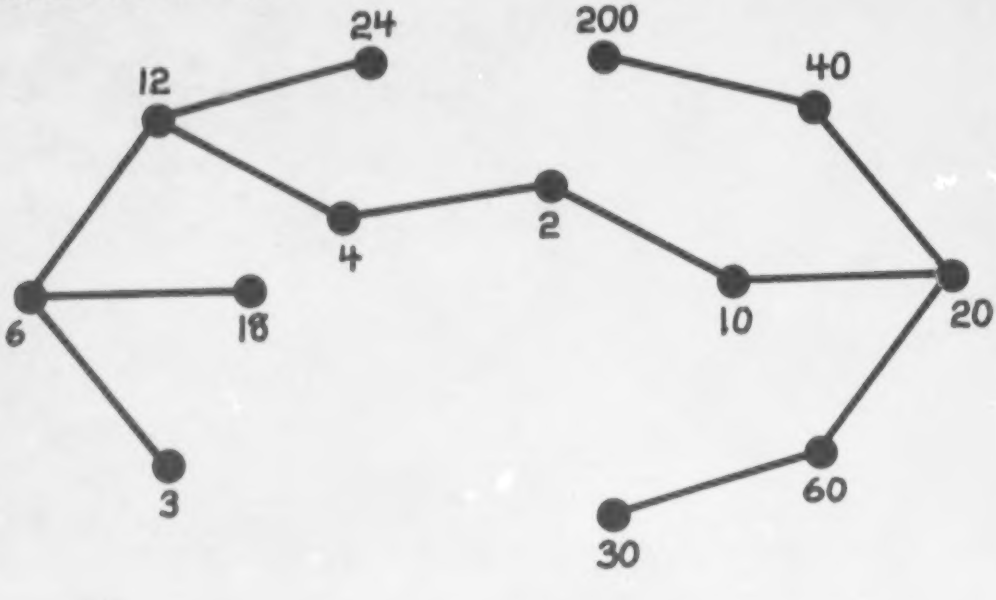
Write this information on the board.

prime factor relation



T: Let's build a road between 6 and 20. How should I start?

Accept any student suggestion as to which cords and numbers should be added to the picture. Make this an open exercise with no wrong answers unless the prime factor relation rule is violated. Eventually, your class is likely to find several roads between 6 and 20, perhaps a shortest road. In the meantime, encourage the students to freely explore the situation. Of course, if certain attempted roads become too long or difficult, accept student suggestions to start again either from 6 or from 20. For example, the completed picture might look like the one below.



L 14

T: How long is the road we found between 6 and 20?

S: 5.

If student interest is high, invite them to extend the picture and find another road between 6 and 20, perhaps a shorter road. If a shortest road of length 3 is found, write "pfd (6,20) = 3" on the board.

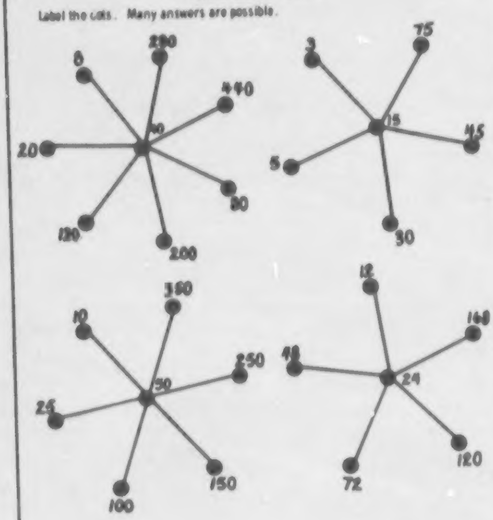
Distribute Worksheets L21 and L22. Students who complete these worksheets quickly could be given a couple problems from Exercise 3A. After a while, copy one or two problems from Worksheet L22 on the board and invite students to present at least one solution for each problem.

Name ______________________

L 21 *

PRIME FACTOR RELATION

Label the dots. Many answers are possible.


8
280
440
40
20
80
120
200
3
75
15
45
5
30
10
350
250
50
25
100
150
12
168
48
24
120
72

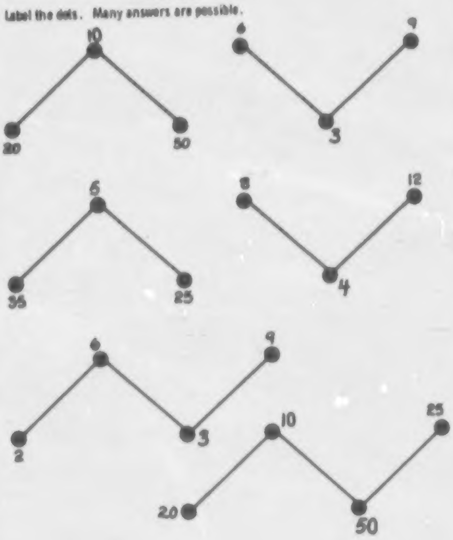
(Many solutions are possible.)

Name ______________________

L 22 **

PRIME FACTOR RELATION

Label the dots. Many answers are possible.


10
20
50
6
9
3
6
35
25
8
12
4
6
9
2
3
10
25
20
50

(Many solutions are possible.)

L 15

3.5 Prime Factor Distance #2

CAPSULE LESSON SUMMARY

Review the definition of the prime factor relation and label dots in a picture involving the prime factor relation. Build roads using the prime factor relation, and find the prime factor distance between some pairs of whole numbers. Determine all the whole numbers less than 50 which are at a prime factor distance of 2 from 10.

MATERIALS

Teacher :	Colored chalk
Student :	Paper ; colored pencils

DESCRIPTION OF LESSON

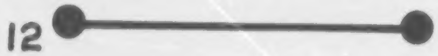
If it is feasible, draw the picture for Exercise 1 on the board before the lesson begins.

Exercise 1

Ask the class to name all the positive primes smaller than 30. Write them on the board.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Write "PRIME FACTOR RELATION" on the board and draw a red cord.



T: The red cord is for the prime factor relation. The dots are for whole numbers. Do you remember the rule for this relation?

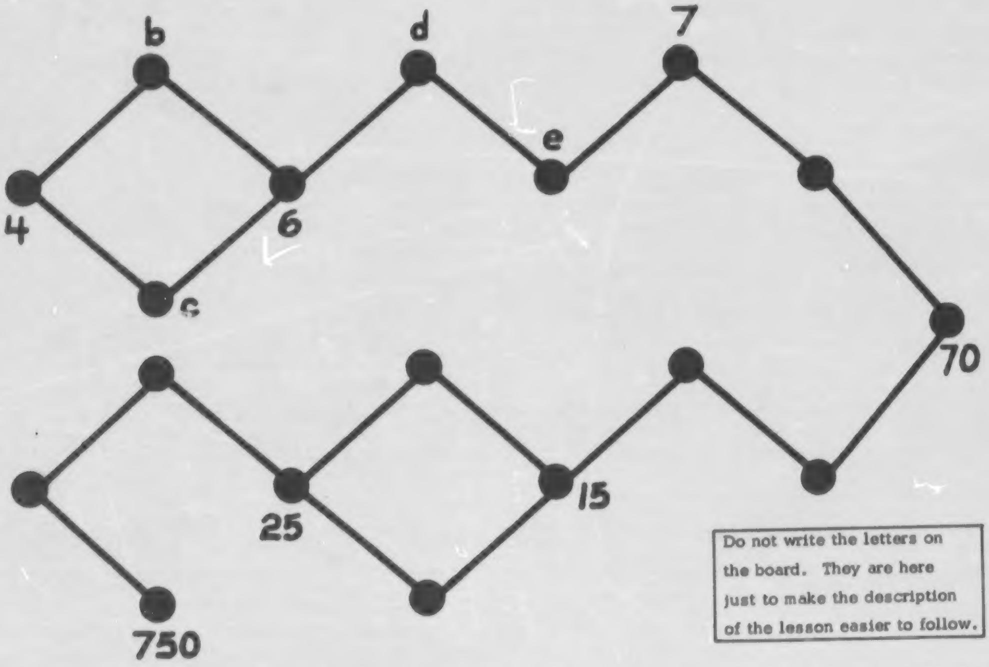
S: Two whole numbers are joined by a red cord if and only if one of the numbers equals a positive prime number times the other number.

T: Which numbers could be joined to 12 with a red cord?

S: 4, because 3 × 4 = 12 and 3 is a positive prime number.

S: 60, because 5 × 12 = 60 and 5 is a positive prime number.

Ask for a few more numbers which could be joined to 12. Then draw this picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The dots are for whole numbers. The red cords are for the prime factor relation. Which number could this be (point to b)?

S: 2.

T: Why?

S: 2 × 2 = 4, 3 × 2 = 6, and both 2 and 3 are positive prime numbers.

Label b "2".

T (pointing to c): Which number could this be?

S: 12, because 2 × 6 = 12, 3 × 4 = 12, and both 2 and 3 are positive prime numbers.

Label c "12".

T (pointing to "6" and "7"): We must build a road between 6 and 7 with three cords. Remember that each dot in an arrow picture must be for a different number. How could we start?

Accept students' suggestions until a solution is found. The class discussion might develop along these lines.

S (pointing to d): This could be 18, because 3 × 6 = 18 and 3 is a positive prime number.

T (pointing to d): If 18 were here, could we build a road from 18 to 7 with just two cords?

When the class realizes that this is impossible, erase "18".

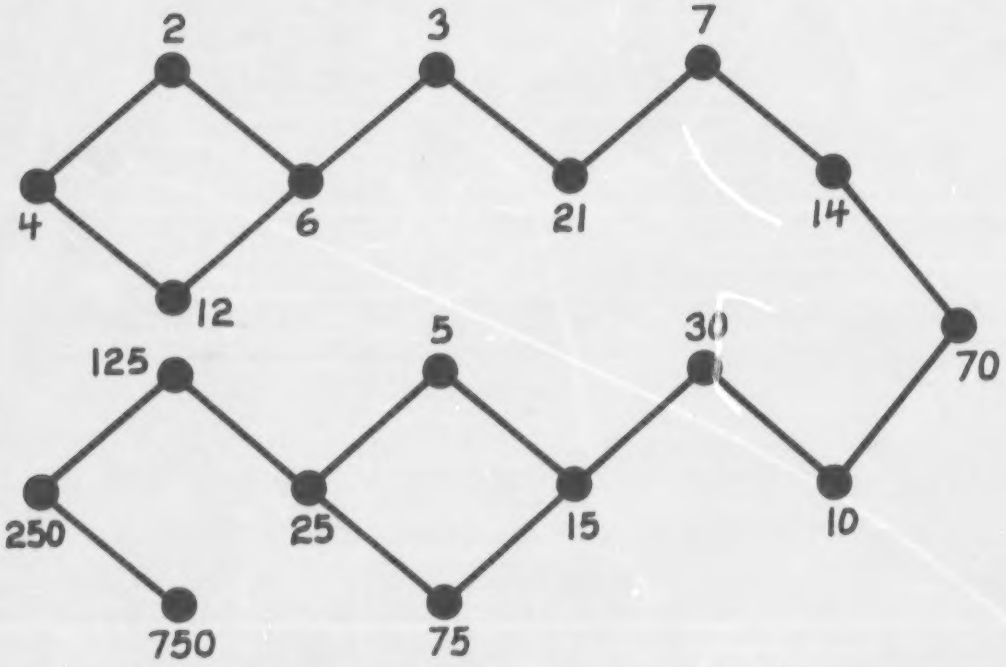
T (pointing to d): Which number could this be?

S: 2.

S: No, another dot is already labeled "2".

S: That number (d) could be 3 and the next number (e) could be 21.

Continue in this manner until all the dots are labeled. Insist that each dot be for a different number. Your arrow picture might look like the following illustration at the completion of this activity.


2
3
7
4
6
21
14
12
70
125
5
30
250
25
15
10
750
75

T: What is the prime factor distance between 15 and 25?

S: 2.

T: How do you know that?

S: There are two red cords in a shortest road between 15 and 25.

Write "pfd (15,25) = 2" on the board.

T: What is the prime factor distance between 6 and 7?

S: 3.

T: Do you think you could build a shorter road between 6 and 7?

S: No.

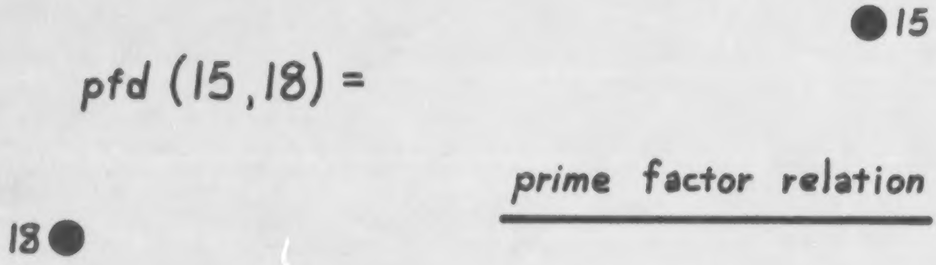
Write "pfd (6,7) = 3" on the board.

183

L 15

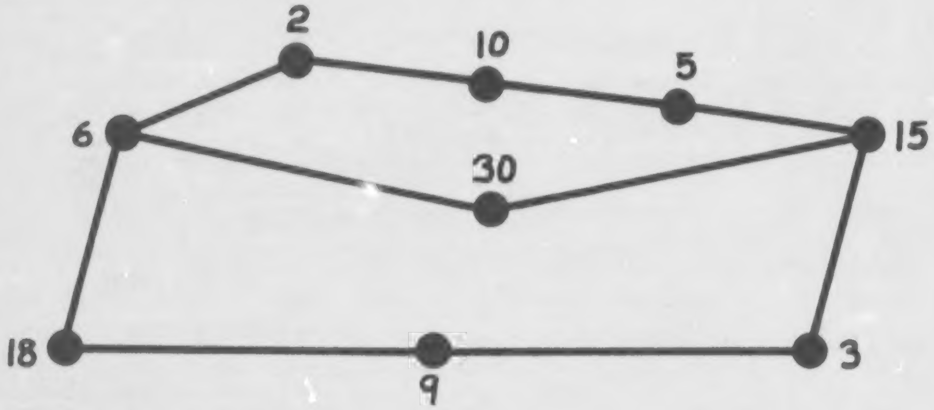
Exercise 2

Write this information on the board.



T: On your paper, use red cords to build a road between 15 and 18. Try to build a shortest road in order to find the prime factor distance between 15 and 18.

Observe the class' work for a few minutes. After many students have built a road between 15 and 18, invite several students to draw their solutions on the board. Do not insist that only shortest roads be drawn. Many roads are possible. For example, your class might draw this picture.



T: What is the length of your shortest road between 15 and 18?

S: 3.

T: Did anyone find a shorter road?

S: No.

Write "pfd (15, 18) = 3" on the board.

If most of your students were unable to build any road between 15 and 18, continue with Exercise 2A. Otherwise, write these problems on the board.

pfd (10,12) =

pfd (9,10) =

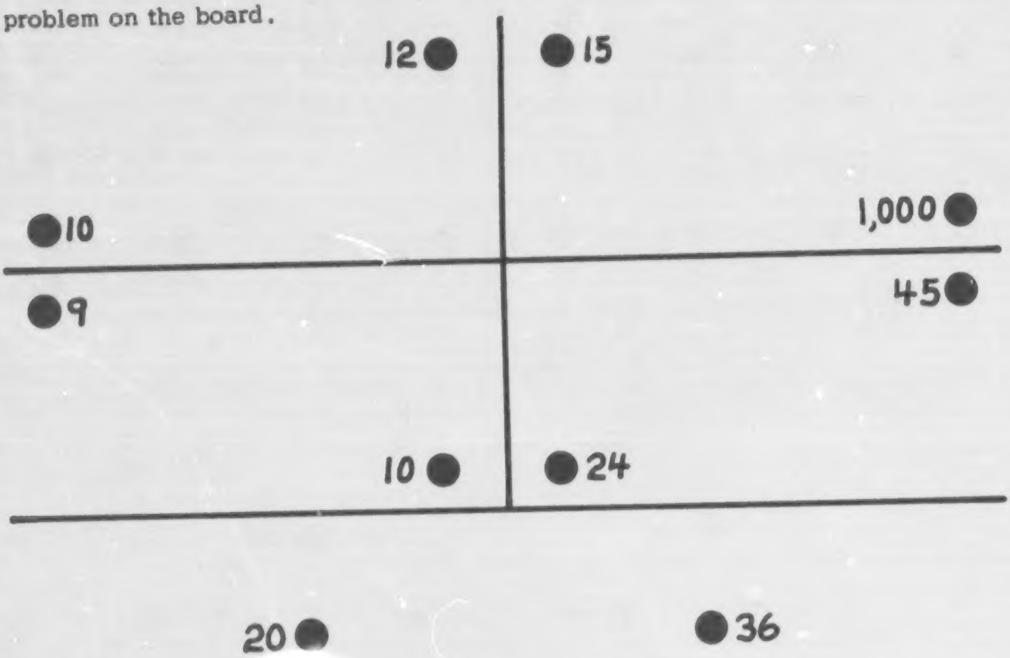
pfd (20,36) =

pfd (15,1000) =

pfd (24,45) =

T: For each of these problems, find the prime factor distance by drawing a shortest road between the two numbers.

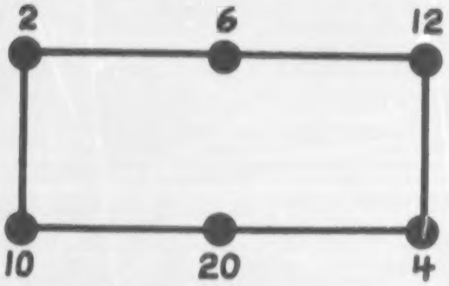
As the students work independently, draw the appropriate pair of dots for each problem on the board.



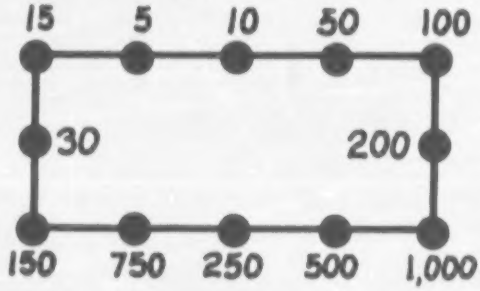
L 15

After most students have built at least two roads, invite students to draw their roads on the board. Encourage students to draw more than one road for each problem. Do not insist that only shortest roads be drawn, but let students who find shorter roads add them to the picture. After a while, ask for the prime factor distance between each pair of numbers. The prime factor distance and two of the shortest roads for each problem are presented below. Many other roads are possible.

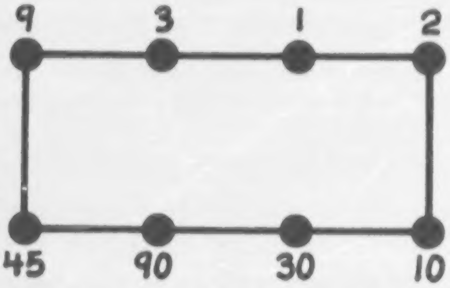
pfd (10,12) = 3



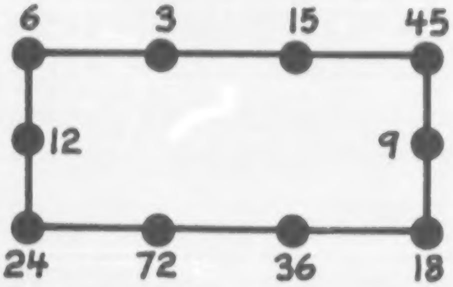
pfd (15,1000) = 6


15
5
10
50
100
30
200
150
750
250
500
1,000

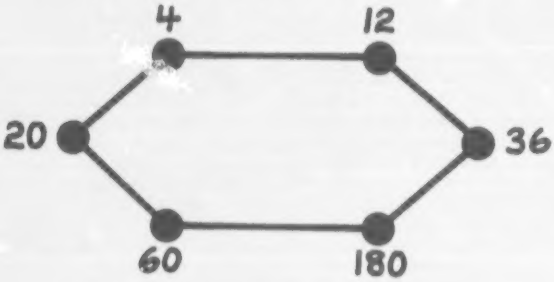
pfd (9,10) = 4


9
3
1
2
45
90
30
10

pfd (24,45) = 5


6
3
15
45
12
9
24
72
36
18

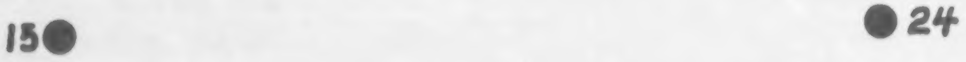
pfd (20,36) = 3


4
12
20
36
60
180

Continue with Exercise 3.

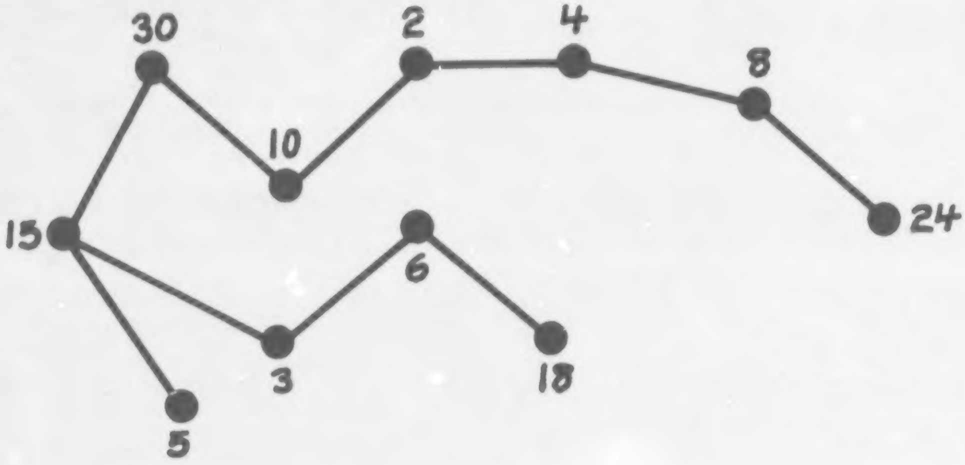
Exercise 2A (This exercise is only for a class having difficulty in Exercise 2.)

Erase the board and write this information.



T: Let's build a road between 15 and 24. How should I start?

Accept student suggestions of which cords and numbers should be added to the picture. Encourage all the students to participate in this activity. Do not comment on the quality of the suggestions since many roads are possible. Let the students start again from either 15 or 24 if they wish. Eventually the class will build a road between 15 and 24; for example, the completed picture might be similar to the picture below.



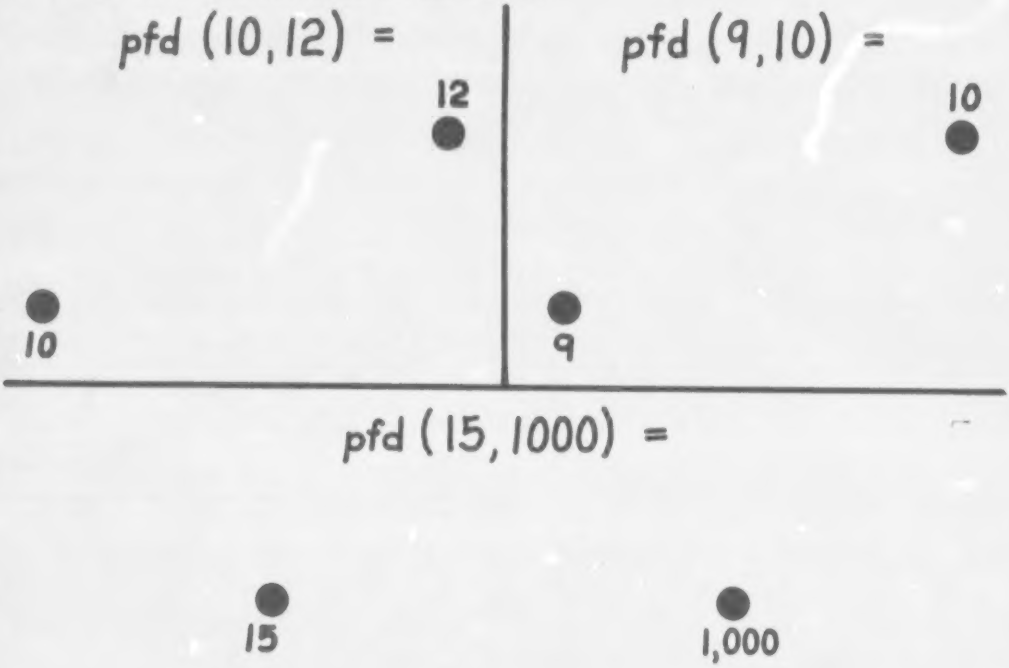
T: What is the length of our road between 15 and 24?

S: 6.

L 15

If interest is high, ask the class to find another road, perhaps a shorter one, between 15 and 24. If a shortest road of length 4 is found, write "pfd (15,24) = 4" on the board.

Write the following problems on the board.



T: For each problem, write the two numbers on your paper and try to build a road between the two numbers.

Let the students work independently for a while. Assist any students who have difficulty building the roads. After a while, invite students to draw their roads for each problem on the board. Do not insist that only shortest roads be drawn, but let students who find shorter roads add them to the picture. Then discuss the prime factor distance between each pair of points. The prime factor distance and two of the shortest roads for each of these problems are presented at the end of Exercise 2.

Continue with Exercise 3 only if there is sufficient time.

Exercise 3

Write this information on the board.

$$pfd\,(10, Flip) = 2$$
$$Flip < 50$$

T: Flip is a whole number smaller than 50. The prime factor distance between Flip and 10 is 2. Which numbers could Flip be?

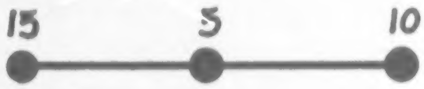
Adapt the following discussion to your student's suggestions.

S: Flip could be 15.

T: Why?

S: Draw a red cord from 10 to 5, because 2 x 5 = 10 and 2 is a positive prime number. Then draw a red cord from 5 to 15, because 3 x 5 = 15 and 3 is a positive prime number.

Draw this picture on the board.



NOTE: Blue dots are used to emphasize the numbers that Flip could be.

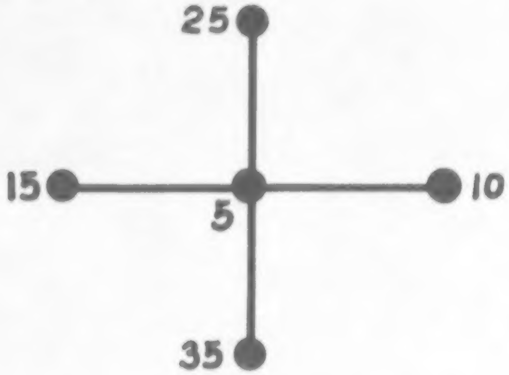
T: What other numbers could Flip be?

S: 25, because 5 x 5 = 25 and 5 is a positive prime number.

S: 35, because 7 x 5 = 35 and 7 is a positive prime number.

L 15

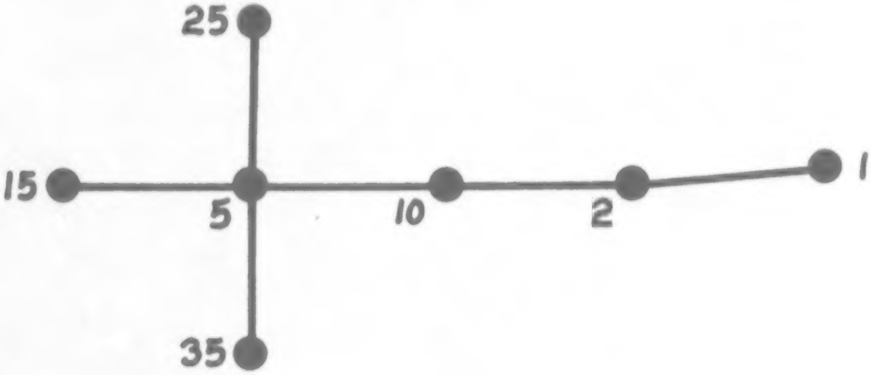
Extend the picture on the board.



T: Are there other numbers that Flip could be?

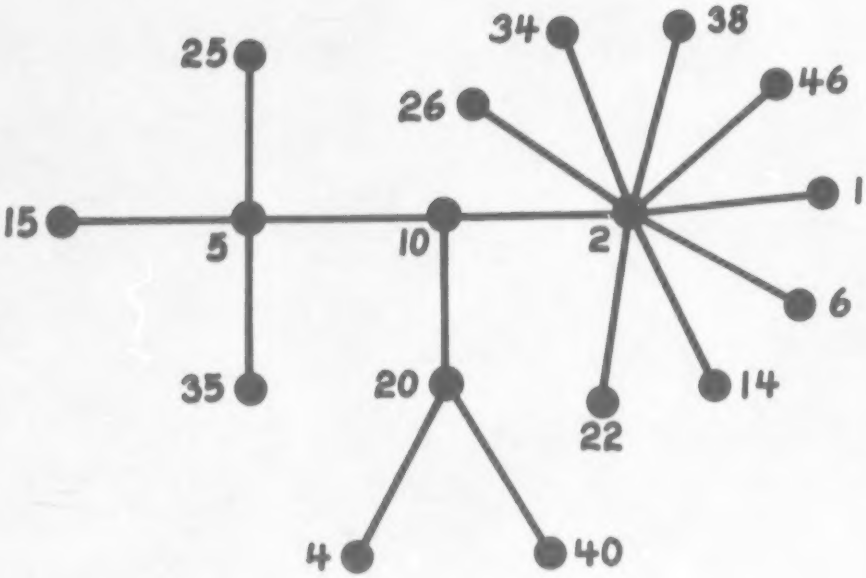
S: Flip could be 1, because you can draw a red cord from 10 to 2 since 5 x 2 = 10 and then draw a cord from 2 to 1 since 2 x 1 = 2.

Add this information to your picture.



After the class has found about four numbers that Flip could be, ask the students to copy the picture from the board and find all the whole numbers less than 50 at a prime factor distance of 2 from 10. Let the students work independently for a while.

Invite students to complete the picture on the board. There are many ways to draw the paths to the thirteen numbers that Flip could be. The completed pictrue might look like the following illustration.


25
34
38
26
46
15
5
10
2
1
6
35
20
22
14
4
40

T: Which numbers could Flip be?

S: 1, 4, 6, 14, 15, 22, 25, 26, 34, 36, 38, 40, and 46.

APPENDIX: The String Game

The String Game is introduced in Part I (Grade Two) of the CSMP Upper Primary Grades Program and is used in many versions throughout the CSMP Elementary School Program. It gives students an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time it involves them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played using A-Blocks, but later on variations which make use of numbers make their appearance. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks or using numbers, and we begin the process of detailing the various possibilities open to you when you play the game. The different variations of the game described in this appendix are referred to in several of the lessons in this chapter. However, we suggest that you do not restrict your playing of the String Game just to the few lessons in this semester when it has been scheduled. It has been found that it is most beneficial (and enjoyable) for students if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

Equipment

PLAYING BOARD

The most easily manageable equipment for this game depends upon your having a magnetic* chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards are not magnetic, try any portable chalkboard (grid board, etc.) that the school has available. If you do not have a magnetic chalkboard available, you can use your regular chalkboard.

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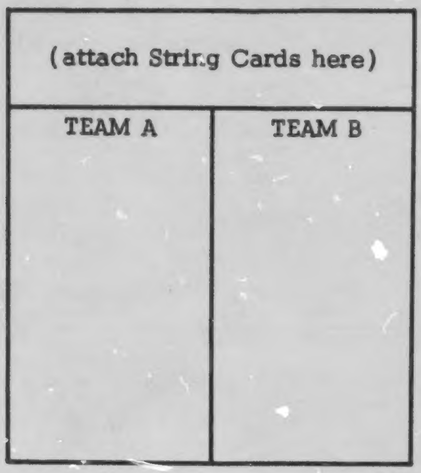
* Strictly speaking the chalkboard itself is not magnetic, but rather magnet-sensitive; i.e., magnets adhere to it. However, for simplicity we call such boards "magnetic".

192

GAME BOARD

The Game Board is divided into three regions all illustrated below.



a) Magnetic: If you do locate a magnetic chalkboard, you will also need a magnetic or metallic Game Board. With a large magnetic chalkboard, say at least three chalkboard squares, the Game Board could be drawn on the board. However, if you have a relatively small (portable) magnetic chalkboard, then you will need to obtain a sheet of metal (minimum size: 60 cm by 80 cm) or locate a convenient metallic surface in the classroom such as the side of a file cabinet on which to put the Game Board. In this case draw the markings for the three regions of the Game Board on a large sheet of (easel pad) paper and tape this paper to your metallic surface.

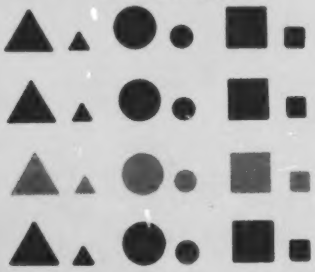
b) Non-magnetic: If you do not have a magnetic chalkboard available for the playing board, then your Game Board could be a large piece of poster board (minimum size: 60 cm by 80 cm). Make the markings for the three regions of the Game Board with a heavy felt-tip pen.

GAME PIECES and STRING CARDS

One set of game pieces and two sets of string cards are needed for each version of the game. One set of the string cards is the "face-up" set and is to be attached to the upper section of the Game Board - it is a constant reminder during the game of the possible labels for the strings. The other set is the "face-down" set.

Game Pieces | **String Cards**

String Game with A-Blocks



△	○	□	RED	BLUE	GREEN	YELLOW	BIG
NOT △	NOT ○	NOT □	NOT RED	NOT BLUE	NOT GREEN	NOT YELLOW	LITTLE

String Game with Numbers: Multiples and Order

-100	-80	-55	-20	-15	-12
-6	-5	-2	-1	0	1
7	10	15	24	27	30
40	45	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5	MULTIPLES OF 10
ODD NUMBERS	LARGER THAN 50	SMALLER THAN 50	LARGER THAN -10	SMALLER THAN -10

String Game with Numbers: Multiples, Divisors, and Order

-100	-80	-55	-15	-10	-5
-1	0	1	2	3	4
5	6	7	8	9	10
12	18	20	24	27	40
45	50	60	99	100	105

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27	LARGER THAN 50	LARGER THAN -10
SMALLER THAN 50	SMALLER THAN -10	ODD NUMBERS	POSITIVE PRIME NUMBERS

Game pieces and string cards for the String Game with Numbers, all versions, can be found in the Numerical String Game Kit. You will have to note which number cards and string cards to use with the particular version you are playing.

a) Magnetic: The game pieces (A-Blocks or number cards) can be magnetized by sticking to the back of each one a small piece of magnetic material. (Magnetic material is included in the Numerical String Game Kit, or it is available in many hobby and notions departments of stores.) To each string card in the "face-down" set, stick a small piece of magnetic material to the front of the card, taking care not to obscure what is written thereon.

b) Non-magnetic: Game pieces and string cards can be attached to the Game Board by means of loops of masking tape stuck to the backs. A "face-down" string card should have a loop of masking tape stuck to the front so that what is written on the card is not obscured. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if a loop of tape loses its stickiness, it can be replaced on the spot. An alternative is to use a small wad of a plastic caulking compound (Rope Caulk or Mortite, for example) in place of the loop of masking tape. This material will maintain its stickiness over a much longer period of time.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the playing board using two (or three) different colors. Next to each of these strings attach one string card from the face-down set. Stand the Game Board on the chalk tray, or have it near and convenient to the playing board. Share the game pieces between the two sections of the Game Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; rows 1, 2, 3 vs. rows 4, 5, 6; etc. Call one team "Team A" and the other "Team B". Before any student takes a turn, correctly place an equal number (at least one) of each team's game pieces on the string picture. This removes from the first players the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by the number of pieces you place on the game board before the game begins.

Object of the Game

Each team tries to place all of its game pieces correctly (according to the face-down string cards) on the string picture. The winning team is the one which first places all of its game pieces correctly and identifies the face-down cards correctly, after playing according to the rules.

Rules of the Game

1) The game is played in silence. This is to avoid one student telling another where to play a certain game piece, or what the strings are for. Each student should have the opportunity to analyze the game alone. Infringement of this rule by anyone is penalized by the talker's team losing its next turn.

2) The teams alternate, and the members take turns within each team. A player comes to the board and selects a piece from his or her team's collection to place in one of the regions of the string picture.

3) You are the judge. If the piece is correctly placed, say "yes". The piece remains on the string picture and the player immediately has a second turn (no player may have more than two consecutive turns). If the piece is incorrectly placed, say "no". The player returns the piece to the team's unplayed collection and play passes to the other team.

   As an aid in judging, prepare a "crib-sheet" showing the correct position of each game piece or at least reminding you of what is on the face-down cards. If at any time you discover that you have made an error, say so immediately and rectify the mistake. Either move an incorrectly placed piece whose position you had approved to its correct region or replace on the string picture a correctly placed piece that had been rejected because you had disapproved its position.

4) When a team has correctly placed all of its pieces, the player who placed the last piece may thereupon attempt to identify each of the string cards.

If these are all correct, the team has won. If a mistake is made (even if it is only in the case of one of the string cards), simply indicate that the identification is incorrect and let the game continue.

5) If a team has exhausted its stock of game pieces, and the strings have not been identified, that team continues on its turn to attempt to identify the strings, while the other team works to place its game pieces.

Version A: The String Game with A-Blocks

This simplest version of the game uses 24 A-Blocks as game pieces and only nine string cards.

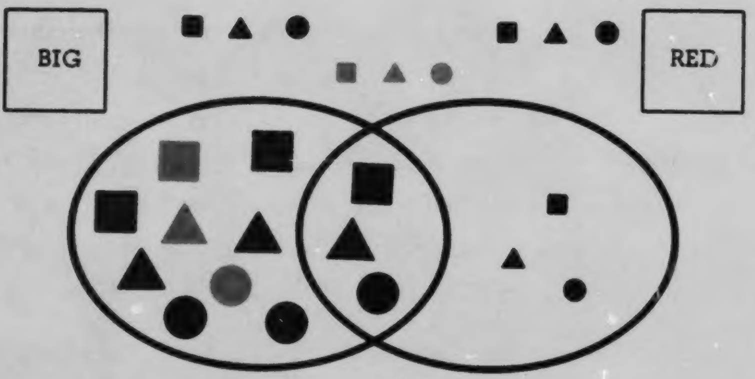
△	○	□	RED	BLUE	YELLOW	GREEN	BIG	LITTLE

(See the description under "Equipment" of game pieces and the two sets of string cards needed for the game.)

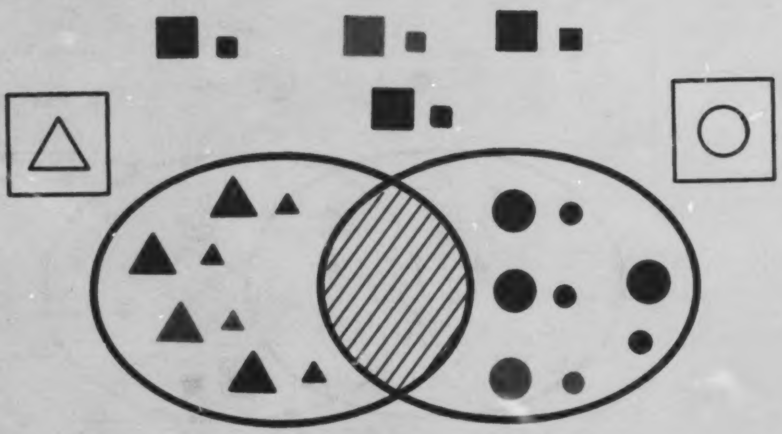
We provide here several example "crib-sheets" for variations of the game with two and three strings.

Variation #1: Two Strings

Example 1: No Empty Regions

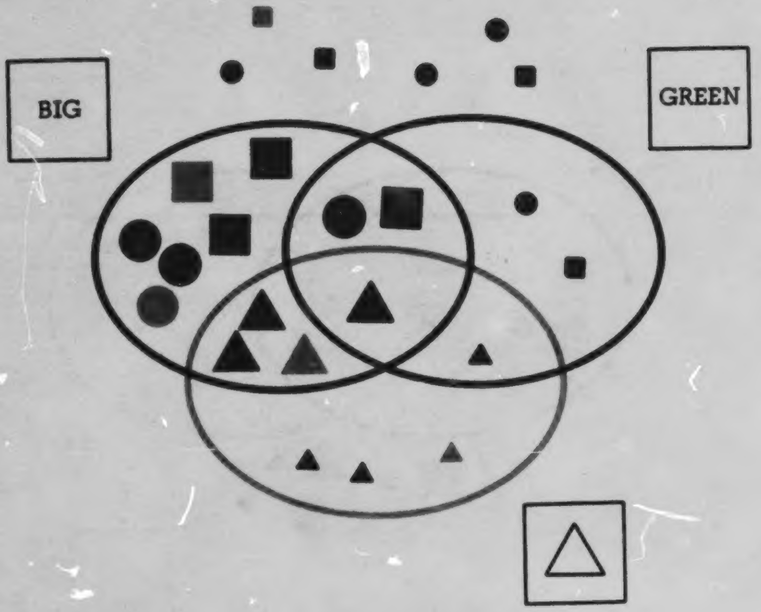


Example 2: One Empty Region

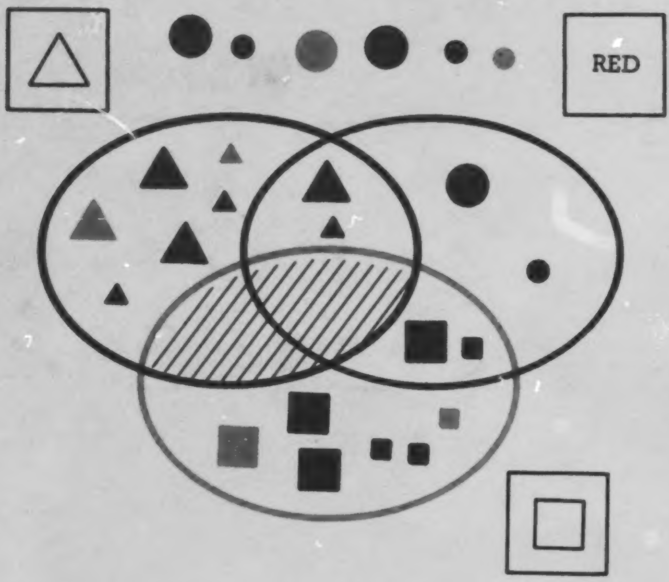


Variation #2: Three Strings

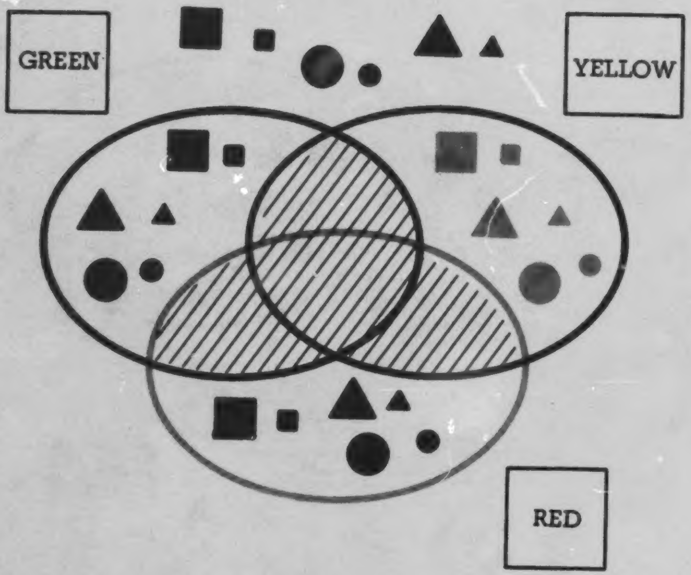
Example 1: No Empty Regions



Example 2: Two Empty Regions



Example 3: Four Empty Regions


GREEN
YELLOW
RED

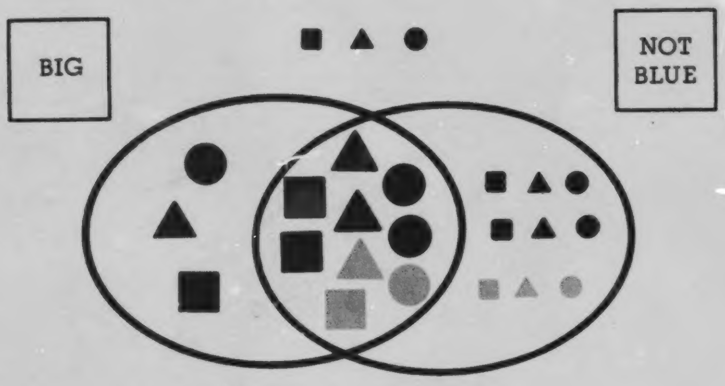
Version B: The String Game with A-Blocks, using "Not" Cards

A more complicated version of the game with A-Blocks uses all 16 of the string cards. (See the description under "Equipment" of game pieces and the two sets of string cards needed for the game with A-Blocks.)

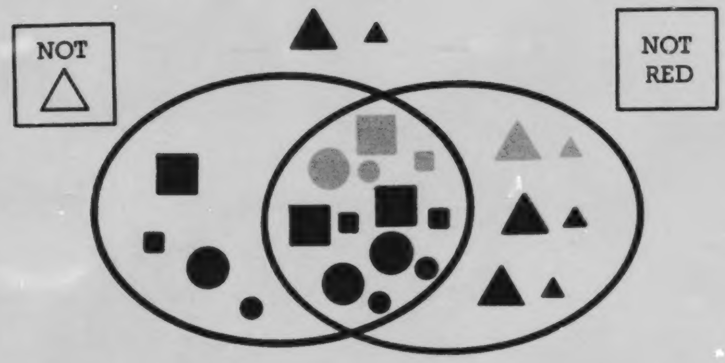
NOTE: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly for "NOT LITTLE" and "BIG"), such a "not" card is redundant. If a student questions why we do not include these possibilities, let the class discuss that it is unnecessary.

We provide here several example "crib-sheets" for this version of the String Game played with two strings.

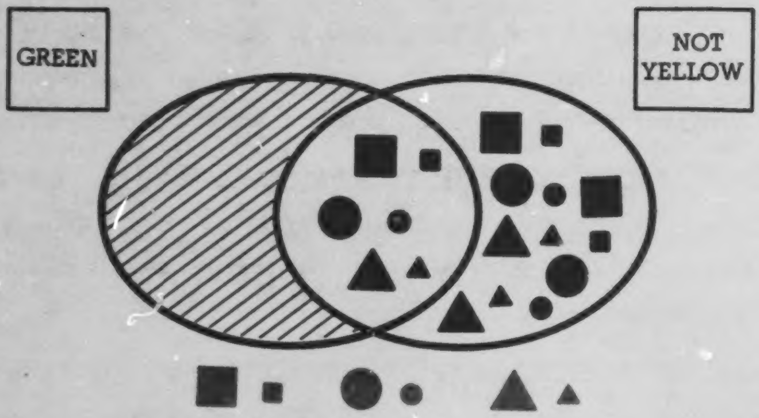
Example 1: No Empty Regions



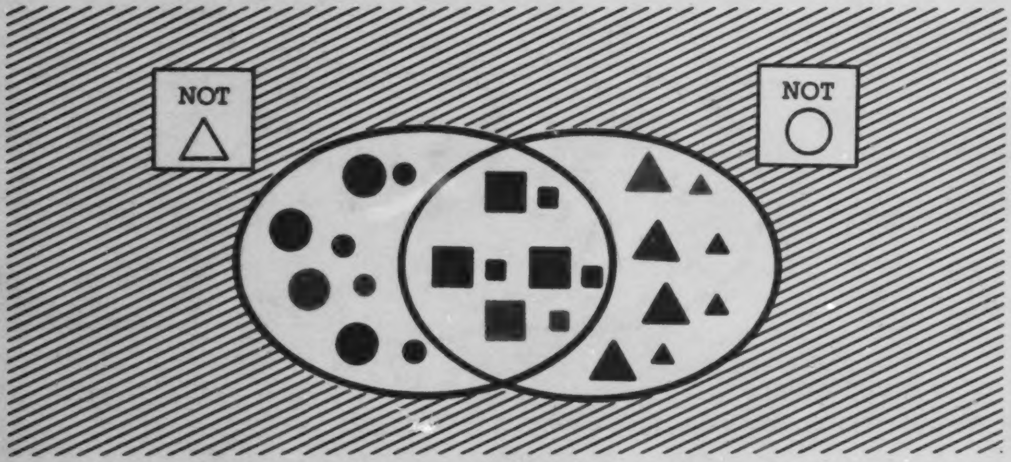
Example 2: No Empty Regions



Example 3: One Empty Region



Example 4: One Empty Region



Of course, by hatching the "outside" region of this diagram we mean to indicate that no game pieces can be placed there correctly. Strictly speaking, that region is not empty because, for example, the number 50 is in the outside region.

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your students.

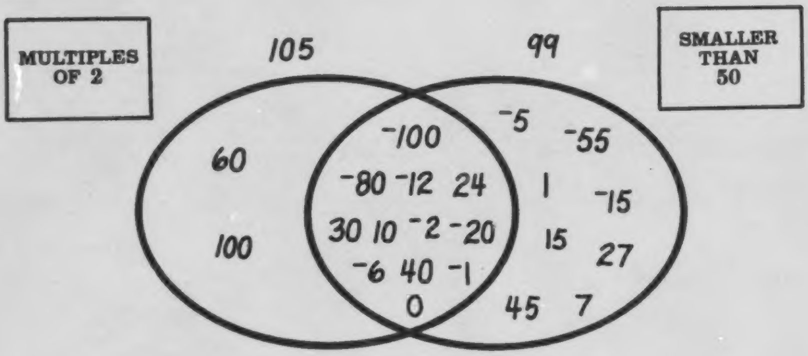
Version C: The String Game with Numbers: Multiples and Order

This version of the game uses 24 number cards as game pieces and two sets of ten string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string.

We provide here several example "crib-sheets" for variations of this game using two and three strings.

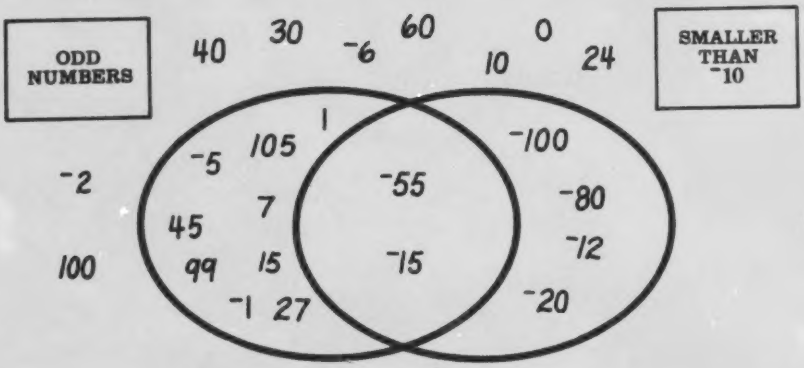
Variation #1: Two Strings

Example 1: No Empty Regions



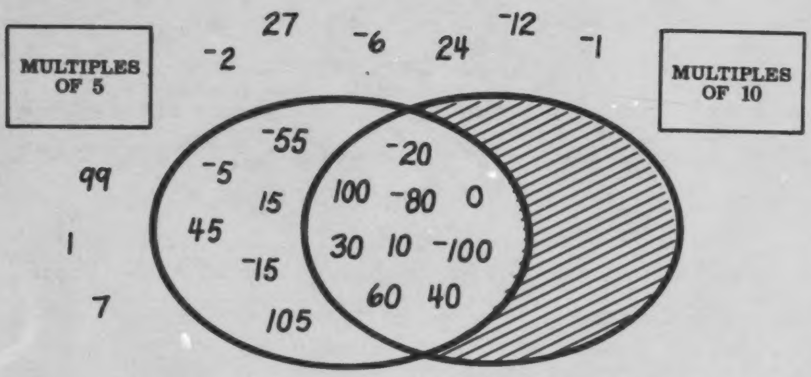
MULTIPLES OF 2
SMALLER THAN 50
105
99
60
100
-100
-80 -12 24
30 10 -2 -20
-6 40 -1
0
-5 -55
1 -15
15 27
45 7

Example 2: No Empty Regions



ODD NUMBERS
SMALLER THAN -10
40 30 -6 60 0
10 24
-2
100
1
-5 105
45 7
99 15
-1 27
-55
-15
-100
-80
-12
-20

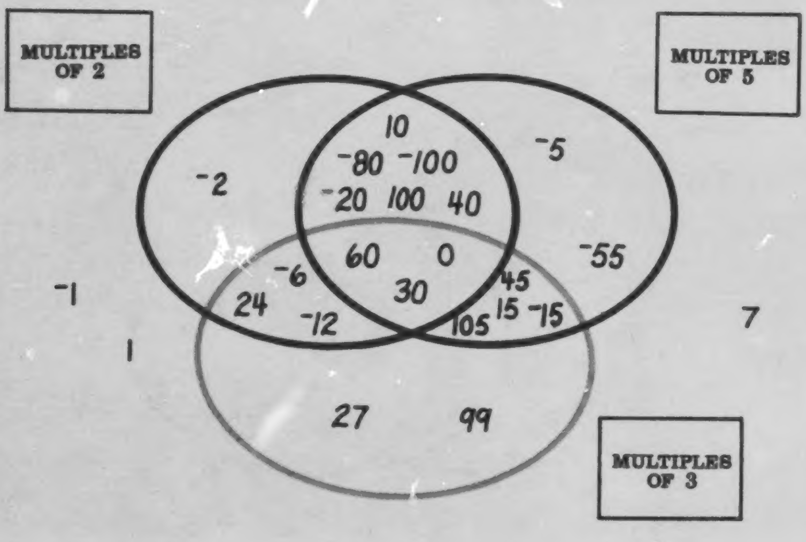
Example 3: One Empty Region


MULTIPLES OF 5
MULTIPLES OF 10
27
⁻6
24
⁻12
⁻1
⁻2
99
1
7
⁻55
⁻5
15
45
⁻15
105
⁻20
100
⁻80
0
30
10
⁻100
60
40

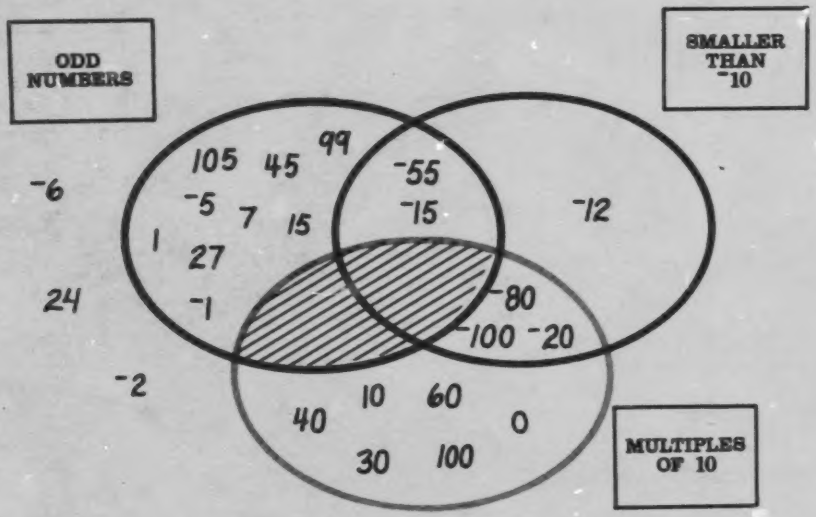
Variation #2: Three Strings

CAUTION: The jump in difficulty from the two-string variation to the three-string variation is much greater in the case of Version C of the String Game than it was in the case of Version A. Hence you should not expect to proceed to Variation #2 in this case as rapidly as you might have done with Version A.

Example 1: No Empty Regions


MULTIPLES OF 2
MULTIPLES OF 5
MULTIPLES OF 3
10
⁻80
⁻100
⁻20
100
40
⁻2
⁻5
⁻55
60
0
30
⁻6
24
⁻12
45
15
⁻15
105
⁻1
1
7
27
99

Example 2: Two Empty Regions

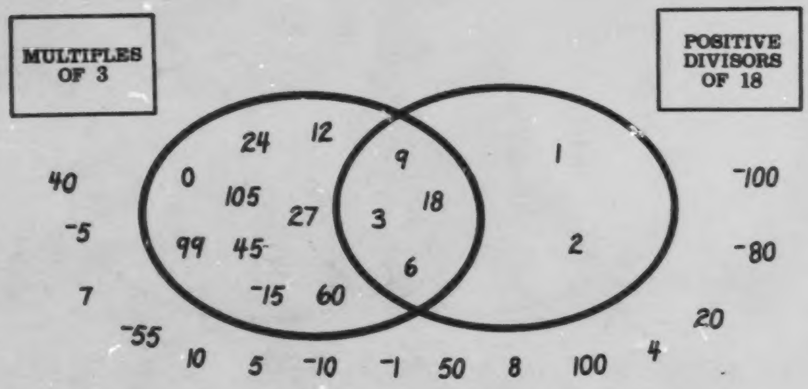

ODD NUMBERS
SMALLER THAN ⁻10
MULTIPLES OF 10
105
45
99
⁻5
7
15
1
27
⁻1
⁻55
⁻15
⁻12
⁻80
⁻100
⁻20
⁻6
24
⁻2
40
10
60
0
30
100

Version D: The String Game with Numbers; Multiples, Divisors, and Order

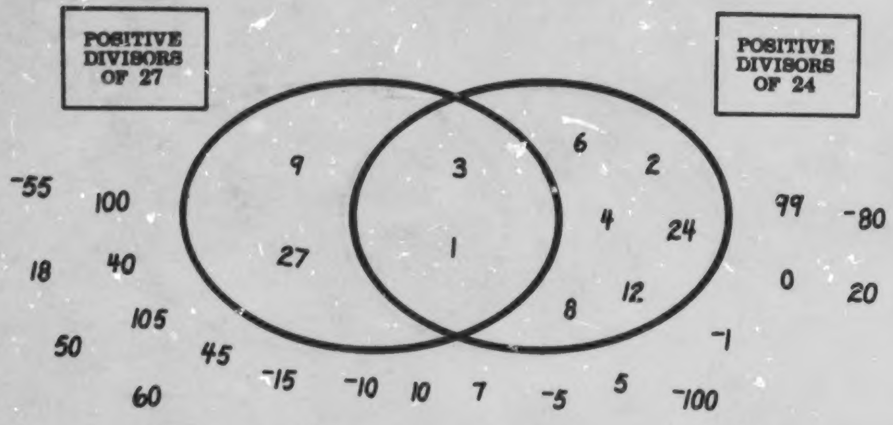
This version of the game uses 30 number cards as game pieces and two sets of 16 string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string. Two examples of this are provided in "crib-sheet" examples provided here; however, before using such a game the players should have previously understood that this was a possibility.

Variation #1: Two Strings

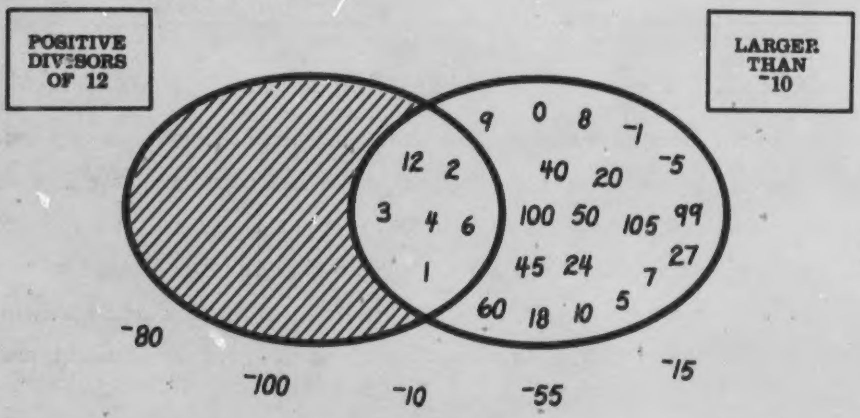
Example 1: No Empty Regions


MULTIPLES OF 3
POSITIVE DIVISORS OF 18
24
12
0
105
27
99
45
⁻15
60
9
18
3
6
1
2
40
⁻5
7
⁻55
⁻100
⁻80
20
10
5
⁻10
⁻1
50
8
100
4

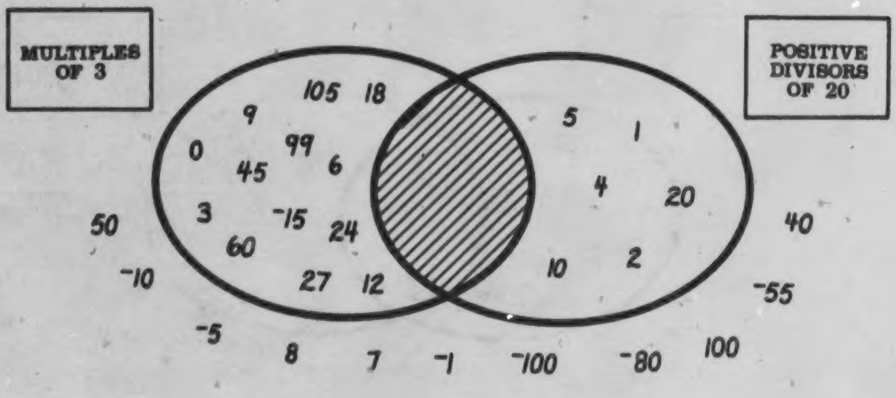
Example 2: No Empty Regions


POSITIVE DIVISORS OF 27
POSITIVE DIVISORS OF 24
9
27
3
1
6
2
4
24
12
8
-55
100
18
40
105
50
45
60
-15
-10
10
7
-5
5
-100
-1
99
-80
0
20

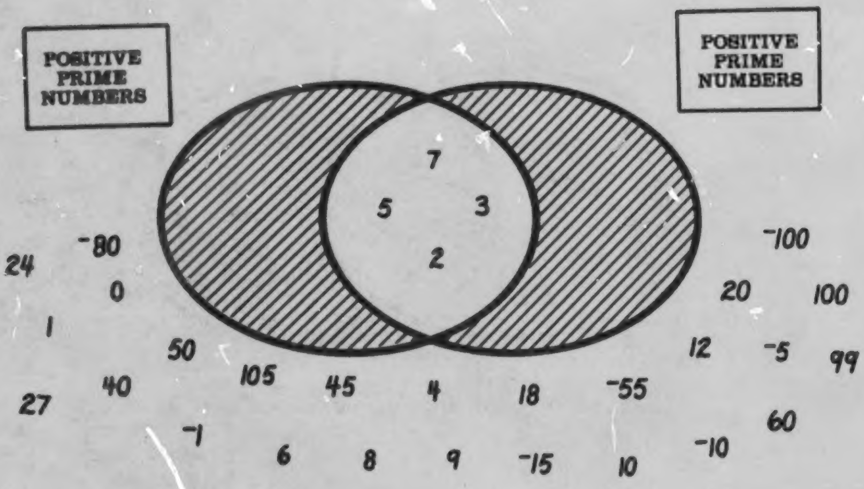
Example 3: One Empty Region


POSITIVE DIVISORS OF 12
LARGER THAN -10
12
2
3
4
6
1
9
0
8
-1
40
20
-5
100
50
105
99
45
24
27
7
60
18
10
5
-80
-100
-10
-55
-15

Example 4: One Empty Region


MULTIPLES OF 3
POSITIVE DIVISORS OF 20
105
18
9
0
99
45
6
3
-15
24
60
27
12
5
1
4
20
10
2
50
-10
40
-55
-5
8
7
-1
-100
-80
100

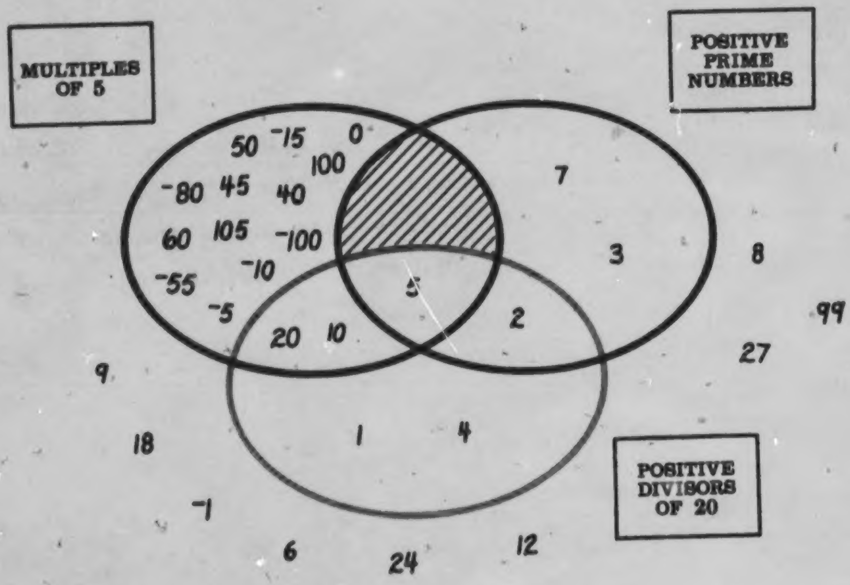
Example 5: Two Empty Regions


POSITIVE PRIME NUMBERS
POSITIVE PRIME NUMBERS
7
5
3
2
24
-80
0
1
50
40
27
-1
105
45
4
6
8
9
18
-55
-15
10
-100
20
100
12
-5
99
60
-10

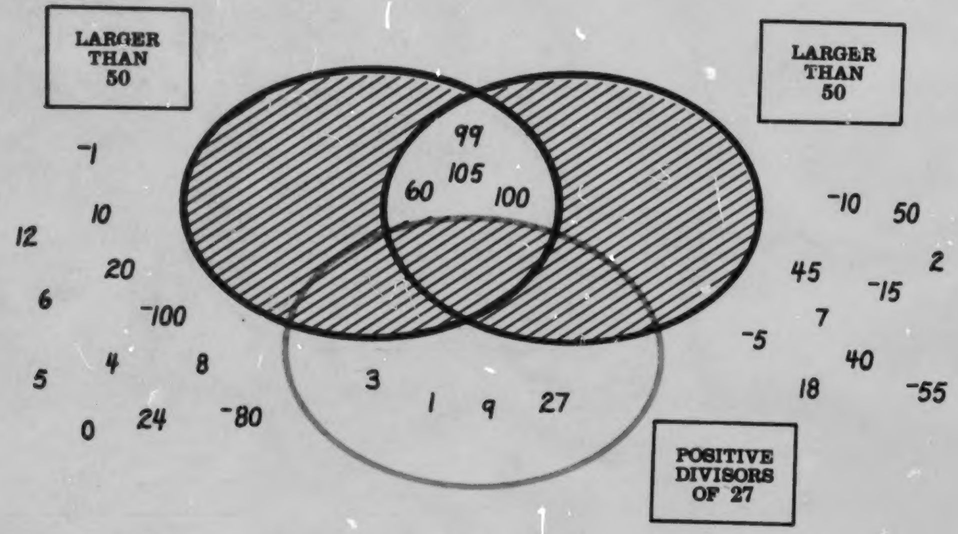
Variation #2: Three Strings

The warning concerning the jump from the two-string variation to the three-string variation on page 138 applies even more so to Version D.

Example 1: One Empty Region


MULTIPLES OF 5
POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 20
50
-15
0
100
-80
45
40
60
105
-100
-10
-55
-5
20
10
5
7
3
2
1
4
8
99
27
9
18
-1
6
24
12

Example 2: Five Empty Regions



Version E: The String Game with Special Scoring Rules

In this version of the game special scoring rules are introduced which determine the winning team. Both the String Game with numbers and with shapes can be played with these scoring rules, however, in the lessons of this book only the Numerical String Game is played this way.

Scoring

Each student is given a list of the possible string labels on a sheet from the individual String Game Analysis Pad. One round of the game is completed by having one player from each team take a turn. After each round the students have an opportunity to privately identify the face-down string cards. They do this by circling on their RED and BLUE lists what they believe is on the hidden tags and submitting it to the judge (teacher). However, each student has just one chance to make a private identification. If a player correctly identifies the string cards after a particular round, then points (according to the following schedule) are added to the player's team score.

Round 1 - 200 points
Round 2 - 150 points
Round 3 - 100 points
Round 4 - 75 points
Round 5 - 50 points
Round 6 - 25 points

After Round 6, play out the game without further private identifications. No more additional points can be scored by a team until the end of the game. The team that correctly identifies the string cards at the end of the game scores an additional 100 points plus 5 points for each unplayed game piece the opposing team has left on the Game Board. The winning team is the one with the greater total score.

Comments

1) Keep count of the rounds played on the board so that both you and the students know which round has just been completed.
2) After each round give the students time (perhaps 15-30 seconds) to study the picture, mark their lists, and if desired to submit them.
3) When the students submit their lists, write on each one the round after which it was turned in and sort them by teams. Do not return them to the students. You can total the team scores at the end of the game. Do not tell the students that their identifications are correct or incorrect, but encourage them to write their findings on a separate sheet of paper so as not to make mistakes when they play. They should continue to check their identifications during the remainder of the game.
4) It may be even more important to insist on silence during this kind of game, because an individual can have more influence on a team's winning or losing. We suggest you be strict about penalties for talking during the game.

Purpose

Scoring rules should not be imposed on the String Game until students have become very familiar and experienced in the play of the game. Many students will already be playing strategically, analyzing the clues given by correctly placed game pieces. Others may still be playing in a random way. The introduction of scoring rules will obviously favor the better students, and perhaps challenge them in a way the usual game is no longer doing. It provides them the opportunity to more dramatically affect their team's chances of winning. For many students the scoring rules may provide incentive to begin analyzing the game and thinking about what information is given by a correctly played game piece.

We do not suggest that you play the String Game very often with the scoring rules. They do impose complication and give perhaps too much advantage to the better players. After playing the game with scoring rules a couple times, you should find that many students have become significantly better players. Returning to the usual way to playing the game, most students will continue to to be better players.

One advantage of using scoring rules when you play the String Game and have only a small amount of time is that you can stop the game before the end and have a reasonable way to determine the winning team. The winning team would be the one with the larger total number of points from identification of string cards.

GEOMETRY
AND
MEASUREMENT

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades
Part IV

Geometry and Measurement

CEMREL, Inc.

TABLE OF CONTENTS

SUGGESTED CHRONOLOGICAL ORDER OF G - LESSONS v

INTRODUCTION vii

CHAPTER 1: MEASUREMENT 1
1.1 Cubes and Volume #1 3
1.2 Cubes and Volume #2 10
Worksheets G01, G02 18
1.3 Mapping a Cube #1 19
Worksheet G05 25
1.4 Mapping a Cube #2 26
Worksheets G07, G08, G09 30
1.5 Surface Area 31
Worksheet G11 37

CHAPTER 2 PARALLELISM 39
2.1 Purchase Vector Space #1 42
2.2 Purchase Vector Space #2 50
2.3 Purchase Vector Space #3 54
2.4 Purchase Vector Space #4 63
Worksheet G12 66
2.5 Purchase Vector Space #5 67
Worksheet G13 70
2.6 Purchase Vector Space #6 71
Worksheet G14 75

SUGGESTED CHRONOLOGICAL ORDER OF G - LESSONS

Lesson One	G1.1	Cubes and Volume #1	3
Lesson Two	G1.2	Cubes and Volume #2	10
Lesson Three	G2.1	Purchase Vector Space #1	42
Lesson Four	G2.2	Purchase Vector Space #2	50
Lesson Five	G2.3	Purchase Vector Space #3	54
Lesson Six	G1.3	Mapping a Cube #1	19
Lesson Seven	G1.4	Mapping a Cube #2	26
Lesson Eight	G2.4	Purchase Vector Space #4	63
Lesson Nine	G1.5	Surface Area	31
Lesson Ten	G2.5	Purchase Vector Space #5	67
Lesson Eleven	G2.6	Purchase Vector Space #6	71

GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as number and calculations) and physical concepts (such as length, area, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense.

A benefit of early developement of geometric thinking is that it provides alternative opportunities for "slow learners". Children develop number concepts at individual rates. Relative lateness of a child's numerical development often signifies a difficulty in remembering number facts. A common response to this problem is assignment of arithmetic drill, which often serves to confirm the diagnosis of slowness. However, numerical development may not be an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

The lessons of this strand provide an opportunity to explore geometric notions informally. The activities are varied to meet other areas of the program and are diffused over a long period of time. As a natural consequence of their involvement in the activities, the students will develop knowledge and skills.

But the effects of such an informal program must be measured in their appearance in subsequent contexts. No goals for mastery can be posted here; none could be certified. The focus of this strand is experience.

1

MEASUREMENT

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area, volume, and perimeter or for comparison of standard units, the lessons of this chapter provide rich, open-ended experiences which relate measurement to other areas of the mathematics curriculum. Direct experience of the tools and concepts is central. Skills and mastery of measurement tasks should develop naturally.

The measurement lessons of this chapter concentrate on volume and surface area of rectangular boxes. After an introduction to calculating volumes of boxes by multiplication of edge lengths analogous to finding areas of rectangles, the challenge to the students is the construction of boxes with specified volumes. The central role of cubes as units of volume measure warrants their special consideration. A strong connection to the World of Numbers strand occurs in the problem of decomposing positive integers as the sum of cubes of integers.

Students exercise their spatial abilities by studying maps of cubes; that is, arrangements of six squares which can be folded to construct a cube. The mapping problem posed for dice leads to a more detailed consideration of the surface of a cube. This in turn leads naturally to finding surface areas. Extending the mapping technique to boxes provides an effective means of calculating surface areas for rectangular boxes.

The lessons in this chapter are supplemented regularly with pages in the "A Selection of Problems" workbooks.

CAPSULE LESSON SUMMARIES

Section 1.1:	Introduce volume by studying rectangular boxes. Develop the technique of calculating the volume of a box by multiplying three appropriate edge lengths. Construct boxes with specified volumes.
Section 1.2:	Introduce the gram and kilogram as units of weight. Find boxes with a volume of 240 cm^3. Using the idea of volume, decompose integers as sums of cubes of integers. (Worksheets G01, G02)
Section 1.3:	Find the ways in which six squares can be positioned with adjacent sides to form a map of a cube. Cut out and fold such maps to build cubes and to study relations of parts of the cube. (Worksheets G03, G04, G05)
Section 1.4:	Review configurations of six squares which form maps of a cube. For some given configurations of squares, determine whether or not they are maps of a cube. Extend the notion of a map of a cube to that of a die, a cube with faces marked in a special way. Label maps of a cube to create maps of a die. (Worksheets G06, G07, G08, G09)
Section 1.5:	Extend the idea of drawing maps of a cube to rectangular boxes. Use maps of a box to find its surface area. Calculate the volume of the box. Find and make maps of other boxes with the same volume and calculate the surface area of each one. (Worksheets G10, G11)

1.1 Cubes and Volume #1

CAPSULE LESSON SUMMARY

Introduce volume by studying rectangular boxes. Develop the technique of calculating the volume of a box by multiplying three appropriate edge lengths. Construct boxes with specified volumes.

MATERIALS

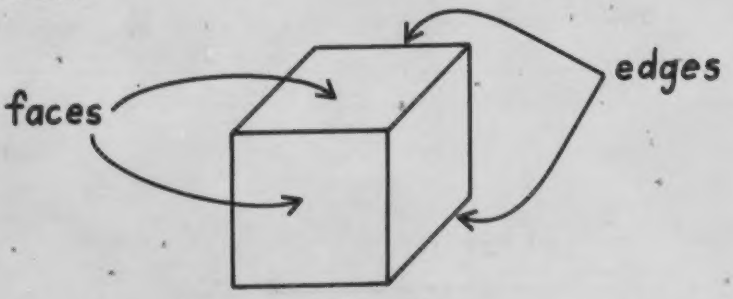
Teacher: Cubes (see note); large box (not cubical)
Student: Paper; pencil

NOTE: (i) You will need two special demonstration cubes. Find or make a cube with edge length exactly one centimeter and a second cube with edge length exactly ten centimeters. These cubes may be constructed of stiff cardboard, but tape the edges well so that there is no gapping and so that the cubes maintain their shape. (ii) Collect thirty cubes, all the same size, to be used for building rectangular boxes. A convenient edge length would be between 2.5 cm and 5 cm. Differing colors among the cubes would make them easier to distinguish from a distance.

DESCRIPTION OF LESSON

Exercise 1

Display a cube.



Conduct a discussion to observe that a cube has special features that distinguish it among all shapes; in particular, that a cube is a box shape for which the six faces are all squares of the same area. In addition, the edges are all of the same length. The discussion, with correct and incorrect responses, might proceed as follows.

G 1

T: What is the shape of this object?

S: Square.

T (indicating one of the faces): One side is a square, but what shape is the entire object?

S: It is a box.

T: But this is a special box.

S: It is a cube.

T: Yes. What is special about a cube?

S: It has parts of the same size.

T: Which parts are the same size? Show us.

S (indicating the faces): All are the same.

T: We call these parts the faces of the cube. How many faces does the cube have?

S: Six.

Write this information on the board.

cube
6 faces

T: And what shape are the faces?

S: All are square.

S: All of them are the same size.

S: All of the faces are squares with the same area.

T: Yes. A cube has six square faces, all of the same area. (Run a finger along an edge of the cube.) This is an edge of the cube. How many edges does a cube have?

S: Twelve.

Write this information on the board.

cube
6 faces
12 edges

Place a large box (which is not a cube) on the desk to serve (in Exercise 2) as a pedestal for displaying box shapes with the colored cubes. Hold up the pedestal box.

T: Is this box a cube?

S: No, a cube has six faces all the same size.

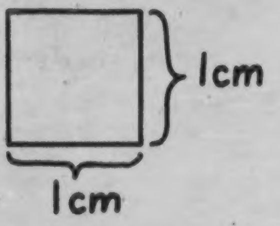
S: Each face of a cube is a square.

Hold up the 1 cm by 1 cm by 1 cm cube.

T: The length of one edge (trace with a finger) of this cube is one centimeter. What is the area of a face (indicate with a finger)?

S: One square centimeter.

Draw a square on the board to represent the face, if there is any difficulty in understanding this fact.



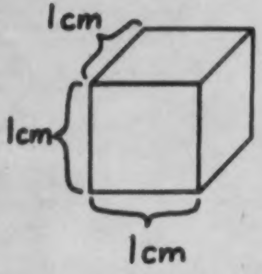
$$\text{area} = 1\ \text{cm}^2$$

T: And what is the area of this face (indicate another face)?

S: Also one square centimeter.

Repeat with the other faces to reinforce the equality of their areas.

T: The amount of space that an object fills is called its volume. Here is a cube that measures exactly one centimeter on each edge. Its volume is one cubic centimeter. There is a short way to write its volume.



$$\text{Volume} = 1\ \text{cubic centimeter} = 1\ \text{cm}^3$$

T: A cube with a volume of one cubic centimeter is really quite small.

Pass your model around the room so that students can gain a sense of the centimeter cube.

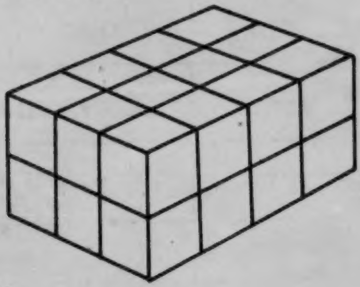
T: The cubic centimeter is a unit that we use to measure volumes.

Exercise 2

Produce your thirty (color) cubes for building boxes. Establish an agreement to treat them as if their volumes were one cubic centimeter (which, of course they are not). This convention is easily accepted by students and is similar to the convention used regularly in working with area on a grid board.

Make a pedestal on your desk with a large box. Construct a sequence of box shapes with the (color) cubes and ask students to calculate their volumes in cubic centimeters. Begin with simple box shapes of a single row of cubes and gradually increase the complexity of the boxes. Aim to develop the technique of calculating the volume of a box by multiplying the lengths of three appropriately chosen edges.

For example, after a few simpler examples, you might construct a box as pictured here and lead the following discussion.



T: What is the volume of this box?

S: There are two layers of cubes with 3 × 4 or 12 cubes in each layer. So there are 2 × 12 or 24 cubes in all, giving a volume of 24 cm^3.

S: I was looking at the box in a different way. There are 2 × 3 or 6 cubes in the front section and four such sections in all. So there are 6 × 4 or 24 cubes in all.

S: There is another way of looking at the box. There are 2 × 4 or 8 cubes along the side. There are three rows like the side, so there are 3 × 8 or 24 cubes in all.

G 1

Ask the responding students to point to the parts of the box as they refer to them.

After a variety of boxes have been considered, ask for the volume of the cube with edge length ten centimeters. (Answer: 1,000 cm^3)

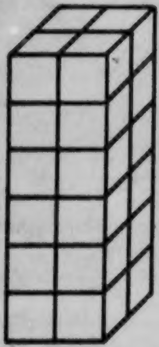
Exercise 3

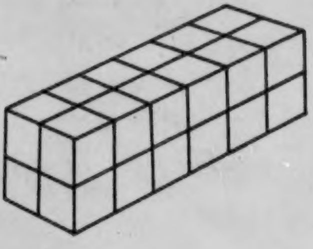
T: Now I will give you a problem. Make a box with a volume of 24 cubic centimeters. You may use 24 of these cubes to make a box. What size boxes can you make?

Display twenty-four of your cubes on the pedestal and invite students to test their ideas. A list of the solutions is given below. Encourage the class to find all of these solutions and list them on the board as they are discovered.

Volume = 24 cm^3	
24 cm by 1cm by 1cm	6cm by 4cm by 1cm
12 cm by 2cm by 1cm	6 cm by 2cm by 2cm
8cm by 3cm by 1cm	4 cm by 3 cm by 2 cm

NOTE: There are alternative descriptions of each of the boxes. For example, "6 cm by 2 cm by 2 cm" is the same box as "2 cm by 6 cm by 2 cm". The only difference is one of orientation. Illustrate this point when such duplications are suggested by constructing the duplications for comparison. For example, the two boxes pictured here are essentially the same.





T: Now suppose that I allow you only twenty cubes to build a box. What size boxes could you make?

Encourage the class to quickly find the solutions.

Volume = 20 cm^3

20 cm by 1cm by 1cm	5 cm by 4 cm by 1cm
10 cm by 2cm by 1cm	5 cm by 2cm by 2cm

T: Now work individually to find what size boxes you can construct with 120 cubes.

Circulate to assist those having difficulty and to check on techniques. At the end of the lesson, collect a list of solutions on the board.

Volume = 120 cm^3

1cm by 1cm by 120 cm	2 cm by 2 cm by 30 cm
1cm by 2cm by 60 cm	2 cm by 3 cm by 20 cm
1cm by 3cm by 40 cm	2 cm by 4 cm by 15 cm
1cm by 4 cm by 30 cm	2 cm by 5 cm by 12 cm
1cm by 5 cm by 24 cm	2 cm by 6 cm by 10 cm
1cm by 6 cm by 20 cm	3 cm by 4 cm by 10 cm
1cm by 8 cm by 15 cm	3 cm by 5 cm by 8 cm
1 cm by 10 cm by 12 cm	4 cm by 5 cm by 6 cm

G2

1.2 Cubes and Volume #2

CAPSULE LESSON SUMMARY

Introduce the gram and kilogram as units of weight. Find boxes with a volume of 240 cm^3. Using the idea of volume, decompose integers as sums of cubes of integers.

MATERIALS

Teacher: Cubes; large box; kilogram weight
Student: Worksheets G01, G02

NOTE: Use the same cubes that were prepared for Lesson G1.1 and the same cardboard box for a pedestal. Obtain a one kilogram weight or fill a small box with sand or the like so that it weights one kilogram.

DESCRIPTION OF LESSON

Exercise 1

Display the ten centimeter cube.

T: The length of an edge of this cube is 10 cm. What is its volume?

S: The cube is 10 cm by 10 cm by 10 cm, so its volume is 1,000 cm^3.

T: Imagine this cube filled with water. The weight of the quantity of water needed to completely fill this cube is called one kilogram. We use "kg" as an abbreviation for "kilogram".

1 kilogram = 1 kg

Pass your sample one kilogram weight around the class to provide a sense of this unit of weight.

Display the one centimeter cube.

T: To weigh small objects, we use another unit. The weight of the water that would completely fill a one centimeter cube is called one gram.

1 kilogram = 1 kg

1 gram = 1 g

T: How many grams are there in one kilogram?

Allow a few minutes for students to decide on an answer. You may need to direct their thinking as suggested by the following response.

S: One gram of water could fill a one centimeter cube. A ten centimeter cube can be built with exactly 1,000 one centimeter cubes, so 1,000 grams of water would fill the ten centimeter cube. Therefore, there are 1,000 grams in one kilogram.

1,000 g = 1 kg

Exercise 2

Set up the large box as a pedestal on your desk. Bring forth the thirty (color) cubes and hold up one of them.

T: Let's pretend again that the length of each edge of this cube is one centimeter. What is the volume?

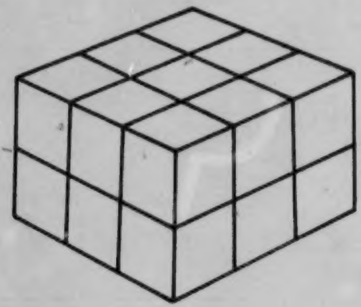
S: One cubic centimeter.

T: I will build some boxes with these cubes and you must find their volumes.

G 2

Construct a variety of rectangular boxes with the cubes on the pedestal. Each time, ask a student to find the volume of the box and to give an indication of a technique used to find it. Reinforce the technique of multiplying the three dimensions of a box to find its volume.

An example of a box and a discussion of it follows.



T: What is the volume of this box?

S: 18 cm^3.

T: How did you find it?

S: I counted nine cubes on the top layer and added another nine cubes for the bottom layer.

S: I calculated "3 x 3 x 2" to get 18 cm^3.

After the students are again comfortable with the multiplication technique for calculating the volumes of the boxes, continue.

T: If we had 240 cubes here, what size boxes could we build using all 240 cubes?

List solutions on the board as they are suggested and check their correctness by multiplying the dimensions. There are many solutions to this problem. It provides an open opportunity for the students to test their ability to visualize constructing boxes. Continue the exercise only as long as solutions are forthcoming from the students.

Volume = 240 cm^3

1 cm by 1 cm by 240 cm	2 cm by 4 cm by 30 cm
1 cm by 2 cm by 120 cm	2 cm by 5 cm by 24 cm
1 cm by 3 cm by 80 cm	2 cm by 6 cm by 20 cm
1 cm by 4 cm by 60 cm	2 cm by 8 cm by 15 cm
1 cm by 5 cm by 48 cm	2 cm by 10 cm by 12 cm
1 cm by 6 cm by 40 cm	3 cm by 4 cm by 20 cm
1 cm by 8 cm by 30 cm	3 cm by 5 cm by 16 cm
1 cm by 10 cm by 24 cm	3 cm by 8 cm by 10 cm
1 cm by 12 cm by 20 cm	4 cm by 4 cm by 15 cm
1 cm by 15 cm by 16 cm	4 cm by 5 cm by 12 cm
2 cm by 2 cm by 60 cm	4 cm by 6 cm by 10 cm
2 cm by 3 cm by 40 cm	5 cm by 6 cm by 8 cm

Exercise 3

T: Remember that all three dimensions of a cube are equal. The lengths of all of the edges of a cube are the same.

Display the ten centimeter cube, for example, and ask again for its volume.

(Answer: 1,000 cm^3)

T: Here are thirty cubes. What is the largest cube that we can build using these centimeter cubes?

There may be some confusion in the beginning. Of course, the largest cube is 3 cm by 3 cm by 3 cm and has a volume of 27 cm^3. Students might start by constructing boxes using all of the blocks. Accept these solutions for discussion to observe that they are not cubes. Some students will be slow to realize that no cube can be constructed using all thirty of the (centimeter) cubes and that not all thirty cubes need to be used.

When the problem is solved to the satisfaction of most of the class, repeat the problem with several other numbers of centimeter cubes, as follows.

Largest cube using at most 15:	2 cm by 2 cm by 2 cm with volume 8 cm^3
Largest cube using at most 7:	1 cm by 1 cm by 1 cm with volume 1 cm^3
Largest cube using at most 100:	4 cm by 4 cm by 4 cm with volume 64 cm^3

T: Now find the largest cube that we can build with 200 of these one centimeter cubes.

S: 5 cm by 5 cm by 5 cm.

T: What is the volume of that cube?

S: 125 cm^3.

Write the information on the board.

$$200 \rightarrow 125$$

T: How many centimeter cubes are left over?

S: 75.

T: What is the largest cube that can be built with the remaining 75 one centimeter cubes?

S: 4 cm by 4 cm by 4 cm.

T: What is its volume?

S: 64 cm^3.

Add to the information on the board.

$$200 \rightarrow 125 + 64$$

T: How many centimeter cubes are left over?

S: 11.

T: What is the largest cube that can be built with the remaining 11 one centimeter cubes?

S: 2 cm by 2 cm by 2 cm with a volume of 8 cm^3.

$$200 \rightarrow 125 + 64 + 8$$

G 2

T: How many centimeter cubes are left over?

S: 3.

S: We can't build any more cubes.

S: But each one is a cube itself.

$$200 \rightarrow 125 + 64 + 8 + 1 + 1 + 1$$

T: Here is another way to write the volume of a cube. What are the measurements of the cube with volume 125 cm^3 ?

S: 5 cm by 5 cm by 5 cm.

T: We write "5^3" for "5 × 5 × 5"; so $5^3 = 125$. We say "125 is the cube of 5." 64 is also the cube of a number; which number?

S: 4, because 4 × 4 × 4 = 64.

S: Then we write "4^3" for "4 × 4 × 4"; so $4^3 = 64$.

T: What about 8?

S: $2^3 = 8$, because 2 × 2 × 2 = 8 and we write "2^3" for "2 × 2 × 2".

T: What about 1?

S: $1^3 = 1$.

In the course of this discussion add to the information on the board.

$$200 \rightarrow 125 + 64 + 8 + 1 + 1 + 1$$

$$200 = 5^3 + 4^3 + 2^3 + 1^3 + 1^3 + 1^3$$

Worksheets G01 and G02 are available for independent work. If time remains at the end of the lesson, conduct a short discussion to compare solutions. These worksheets could be used for homework.

Name______________________ G 01 *

Find the largest cube that you can build

1) using 85 centimeter cubes.
4 cm by 4 cm by 4 cm (64 centimeter cubes)

2) using 304 centimeter cubes.
6 cm by 6 cm by 6 cm (216 centimeter cubes)

3) using 25 centimeter cubes.
2 cm by 2 cm by 2 cm (8 centimeter cubes)

4) using 113 centimeter cubes.
4 cm by 4 cm by 4 cm (64 centimeter cubes)

5) using 621 centimeter cubes.
8 cm by 8 cm by 8 cm (512 centimeter cubes)

Name______________________ G 02 **

Build the largest cube possible with 1,500 centimeter cubes. Then build the largest cube possible with the left over centimeter cubes.
Keep doing this until no centimeter cubes are left over.

11 cm by 11 cm by 11 cm	1,331 centimeter cubes
5 cm by 5 cm by 5 cm	125 centimeter cubes
3 cm by 3 cm by 3 cm	27 centimeter cubes
2 cm by 2 cm by 2 cm	8 centimeter cubes
2 cm by 2 cm by 2 cm	8 centimeter cubes
1 cm by 1 cm by 1 cm	1 centimeter cube
	1,500 centimeter cubes

$$1{,}500 = 11^3 + 5^3 + 3^3 + 2^3 + 2^3 + 1^3$$

Do the same with 2,045 centimeter cubes.

12 cm by 12 cm by 12 cm	1,728 centimeter cubes
6 cm by 6 cm by 6 cm	216 centimeter cubes
4 cm by 4 cm by 4 cm	64 centimeter cubes
3 cm by 3 cm by 3 cm	27 centimeter cubes
2 cm by 2 cm by 2 cm	8 centimeter cubes
1 cm by 1 cm by 1 cm	1 centimeter cube
1 cm by 1 cm by 1 cm	1 centimeter cube
	2,045 centimeter cubes

$$2{,}045 = 12^3 + 6^3 + 4^3 + 3^3 + 2^3 + 1^3 + 1^3$$

1.3 Mapping a Cube #1

CAPSULE LESSON SUMMARY

Find the ways in which six squares can be positioned with adjacent sides to form a map of a cube. Cut out and fold such maps to build cubes and to study relations of parts of the cube.

MATERIALS

Teacher:	Colored chalk; 10 cm cube; Geometry Poster #1 for IG-IV
Student:	Paper; colored pencils; scissors; envelope; Worksheets G03, G04, G05

NOTE: Geometry Poster #1 for IG-IV shows eleven maps of a cube. They are exactly those configurations of six squares which can be folded to make a cube. Cut out each of the eleven maps to use for folding to construct cubes during the lesson.

DESCRIPTION OF LESSON

Exercise 1

Display the ten centimeter cube.

T: How many faces does the cube have?

S: Six.

T: And how many edges?

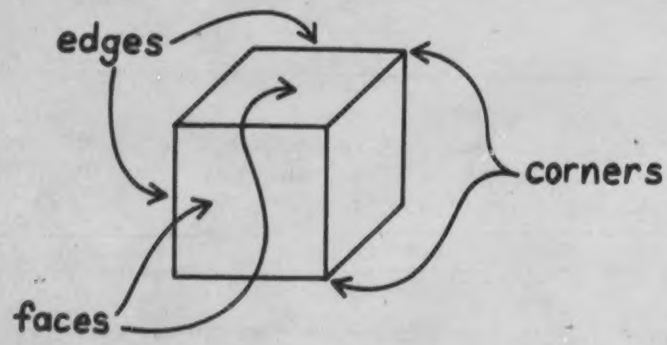
S: Twelve.

Write this information on the board.

cube
6 faces
12 edges

G 6

Indicate several corners on your demonstration cube.



T: What do we call these other important parts of a cube?

S: "Corners".

T: How many corners does a cube have?

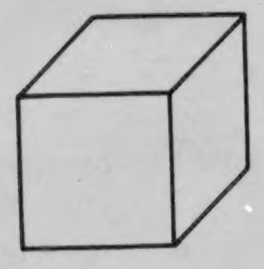
S: Eight.

Add this information to the board.

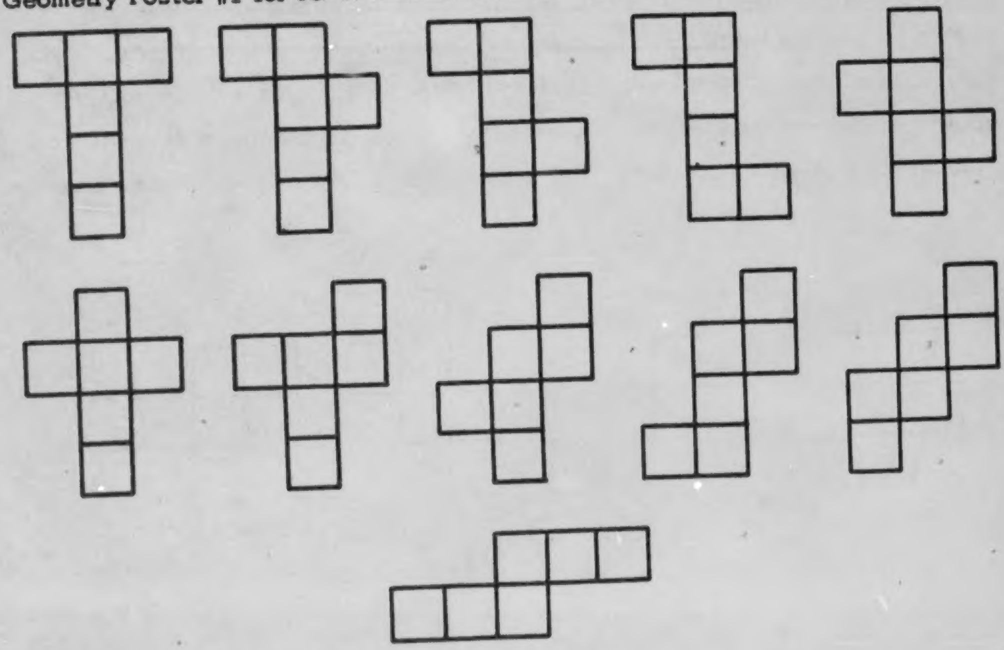
cube
6 faces
12 edges
8 corners

T: Today we will look at a cube more closely. Sometimes to study an object more easily, we draw a map of the object. When we study the earth, for example, we often use a map of the earth. A map is usually designed to show the important features of the object. For a cube, we have a list of the important parts here on the board. Draw a map of the cube on your paper and include these important features.

Each student should now attempt to map a cube on an unlined piece of paper. The instructions are deliberately left vague in order to allow students to search for an effective technique of mapping. Many will simply draw a picture of a cube as follows.



When this solution is offered, copy it onto the board for discussion. The point to be made is that such a picture of a cube does not provide all of the surface of the cube for examination. If after several minutes no alternative solution is suggested, make a remark comparing planar maps of the earth with global maps. Perhaps some students will find a solution by arranging six squares in one of the eleven patterns shown below. Copy such a solution on the board and discuss its construction with emphasis on the way in which it can be folded to make a cube. Produce your copy of the suggested map from Geometry Poster #1 for IG-IV and invite a student to illustrate the folding.

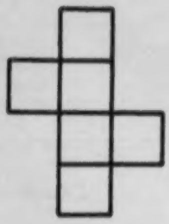


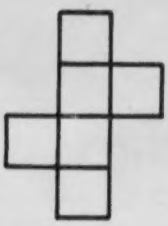
G 6

Distribute Worksheets G03 and G04, scissors, and envelopes for later storage of cutouts. Instruct the students to find other maps of the cube and to test their maps by cutting them out and folding them to form cubes. As solutions are found, draw them on the board and produce your construction paper models for folding.

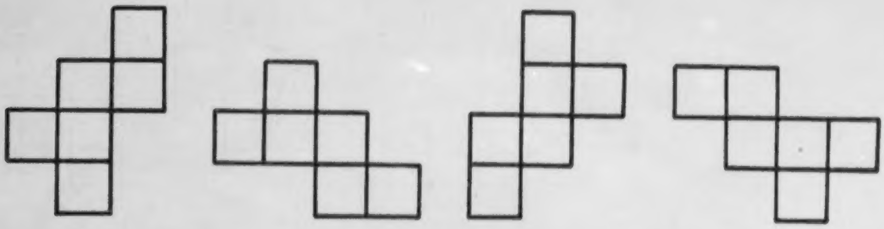
Continue the individual discovery of maps and recording on the board until it is no longer fruitful. It is not necessary that the class discover all of the eleven maps. Reserve ten to fifteen minutes for the next exercise.

NOTE: Students may suggest maps which differ only in orientation.





These two maps can be considered to be the same map for the purposes of this lesson, since one can be changed into the other by simply turning it over. If such a pair is suggested by the students, demonstrate the identification with your construction paper model. Draw the pair of maps on the board and then show that the same paper model can be made to fit each one. Also make no distinctions based on rotation. For example the following maps are to be considered as the same.



Exercise 2

T: How many maps have we found? (Eleven or fewer) What did you have to remember in drawing the maps?

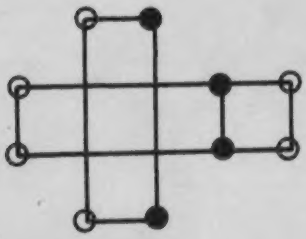
S: There are six faces.

Count the six squares in each of the maps.

T: How many corners does a cube have?

S: Eight.

Invite a student to count the corners in one of the simpler maps. At least two things could happen. The count will be fourteen. In this case, observe that a cube has only eight corners. Ask for an explanation of the discrepency. Lead the discussion, if necessary, to the point that a corner of a cube may appear more than once in a map. In the alternative case, the responding student will count eight by matching the corners spontaneously. Then ask the student to explain how the corners are matched. In either case, develop a system to record the matching of corners. Use colored circles, dots, or crosses to show corners which match. In the end your map might look like this one.

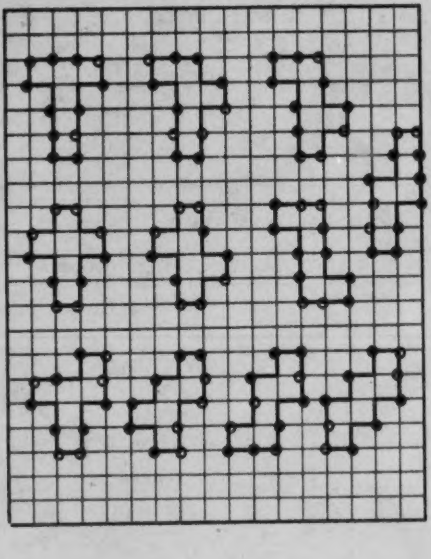


Distribute Worksheet G05. Instruct the students to copy each map from the board and then to mark the corners of a map to show which ones match in the folding. Remind them that the paper models they constructed can be used to help in finding corners that match. If a paper model of a map has not been made by a student, the student should then construct one.

Any student who finishes quickly can look for more maps, if all eleven were not discovered. At the end of this lesson, instruct students to collect their cutout maps of the cube in an envelope to be saved for use in Lesson G1.4. Also, save your poster cutouts.

Name ______________________

G 05



1.4 Mapping a Cube #2

CAPSULE LESSON SUMMARY

Review configurations of six squares which form maps of a cube. For some given configurations of squares, determine whether or not they are maps of a cube. Extend the notion of a map of a cube to that of a die, a cube with faces marked in a special way. Label maps of a cube to create maps of a die.

MATERIALS

Teacher:	10 cm cube; cutouts from Geometry Poster #1 for IG-IV; die
Student:	Set of cutout maps of cube; pencil; Worksheets G06, G07, G08, G09

NOTE: Use the 10 cm cube from Lesson G1.1 and the cutout maps of the cube from Lesson G1.3.

DESCRIPTION OF LESSON

Distribute the students' collections of cutout maps of a cube which were saved from Lesson G1.3 and Worksheet G06 for producing more cutouts.

Exercise 1

T: Today we are going to look at maps of a cube again. Last time you each made some maps of a cube. Let's draw as many of them as we can on the board.

Students should suggest maps from their collections. Draw both correct and incorrect suggestions on the board. Ask students to justify or to eliminate each suggestion by producing a cutout to fold. Tell the students they may use Worksheet G06 to produce cutouts which are not already in their collections. Cross out incorrect maps. When the ideas of the class are exhausted, erase the board drawings and distribute Worksheet G07.

T: Here is a sheet with a variety of configurations of squares. Some of them are maps of a cube and some are not. Find which are correct maps and cross out the others.

Allow five to ten minutes for this exercise. Encourage students to use their cutout maps or to cut out copies of the maps and to check their correctness by folding. At the end of the exercise, check the work collectively.

Exercise 2

Display the die.

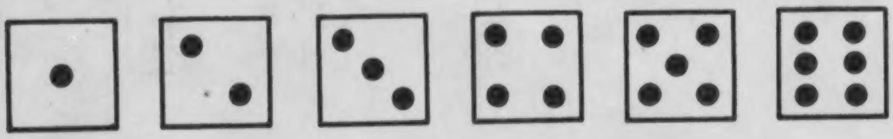
T: This is a die. What shape is it?

S: It is a cube.

T: What is on the faces of a die?

S: There are a different number of dots on each face: one, two, three, four, five, and six.

Draw six squares on the board to illustrate the six faces.



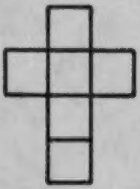
T: Six faces, each with a different number of dots. We have been drawing maps of a cube. Now let's draw a map of this die. What should we do?

S: We already have many maps of the cube. We only need to mark the faces with dots or numbers.

T: Let's try with this map.

G 7

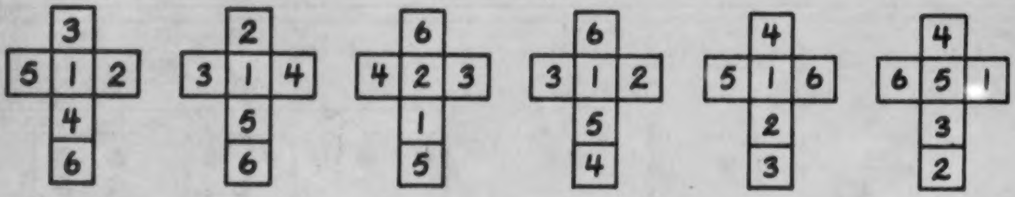
Choose one of the simpler maps, for example,



... and tape your cutout copy on the board.

T: Everyone find or make a cutout copy of this map. Then mark its faces to make a map of the die.

When many students are finished, collect a sample of solutions on the board. Draw a number of copies of the map and solicit labelings from the class. Write numerals instead of dots to ease and speed the labeling. The result will be a variety of labelings similar to those in the following illustration.

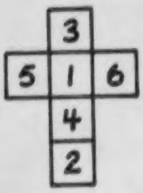


T: We should check that each of these labelings is correct. We already know that each of these is a map of a cube, but is each labeling correct to make a die?

For each map, ask a student to present a cutout with that labeling and to demonstrate its folding to make a die. Since the students' models will be too small to be seen around the room, ask the student to report the result of comparison with the demonstration die. Some of the proposed labelings may be incorrect (as for example, the fourth and fifth ones above). The incorrectness of a labeling will be discovered when a model is folded and compared with the demonstration die. The object of the activity is to discover that the numbers assigned to opposite faces of a die add to 7. Thus "1" and "6" are on opposite faces, "2" and "5" are on opposite faces, and "3" and "4" are on opposite faces.

NOTE: When a map is folded, the numerals may be on the inside surface of the resulting cube. In this case, you can easily redo the folding so that they appear on the outside surface.

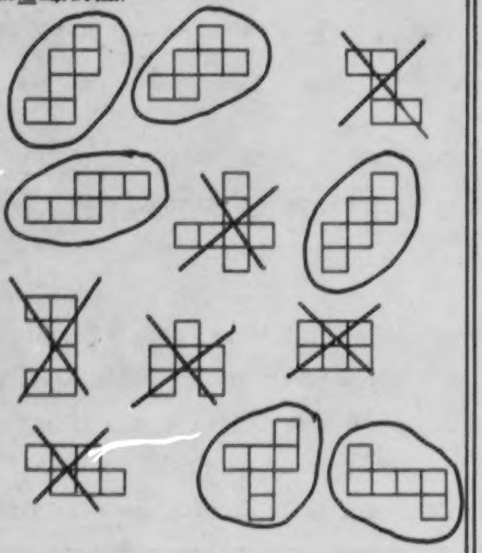
HISTORICAL NOTE: Dice were not always marked in this way. The development of carefully marked and balanced dice is relatively recent in the history of such devices. One result of the modern labeling is that no cycle of four faces is favored for low or high scores in rolling. A die marked as shown below could be carefully rolled along the 3-1-4-2 faces to produce a consistent low score, or it could be rolled along the 3-6-4-5 faces to produce a consistent high score. Such anti-social manipulation of the die is prevented by the now customary labeling. You might label a demonstration die as shown here to demonstrate and discuss its unfairness with the class.



Complete the lesson with individual work on Worksheets G08 and G09. The worksheets show maps of a cube. The instruction is to complete the labelings of the sides to produce a map of a die. Students may be aided in solving these problems by their cutout maps. Encourage students who finish quickly to determine matching corners in the maps on the worksheets in the manner of the last exercise of Lesson G1.3.

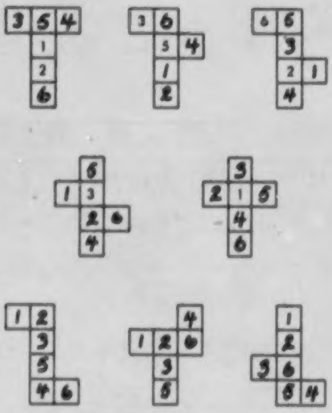
Name ______________________ G 07 *

Circle the configurations which are maps of a cube. Cross out those which are not maps of a cube.



Name ______________________ G 08 **

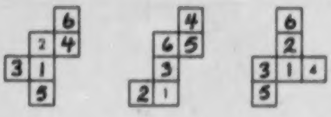
Complete the labeling of each map to make a map of a die.


3 5 4 1 2 6
3 6 5 4 1 2
6 5 3 2 1 4
5 1 3 2 6 4
3 2 1 5 4 6
1 2 3 5 4 6
4 1 2 6 3 5
1 2 3 6 5 4

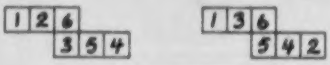
(Other solutions are possible.)

Name ______________________ G 09 ***

Complete the labeling of each map to make a map of a die.


6 2 4 3 1 5
4 6 5 3 2 1
6 2 3 1 4 5

These maps are the same. Label them differently so that both are maps of a die.


1 2 6 3 5 4
1 3 6 5 4 2

(Other solutions are possible.)

1.5 Surface Area

CAPSULE LESSON SUMMARY

Extend the idea of drawing maps of a cube to rectangular boxes. Use maps of a box to find its surface area. Calculate the volume of the box. Find and make maps of other boxes with the same volume and calculate the surface area of each one.

MATERIALS

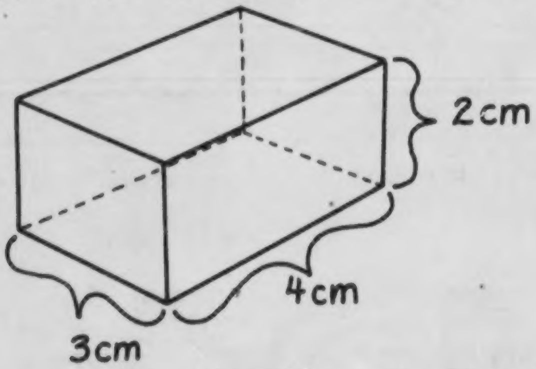
Teacher: Colored chalk; grid board (optional)
Student: Colored pencils; scissors; Worksheets G10, G11

DESCRIPTION OF LESSON

Exercise 1

T: I am thinking about a box. I can draw a picture of it.

Draw a picture of a box on the board and indicate its dimensions.



T: As you can see my box is 2 cm high, 3 cm wide and 4 cm long. My box is colorful, too. It is painted with three colors: red, blue, and yellow. How many faces does my box have?

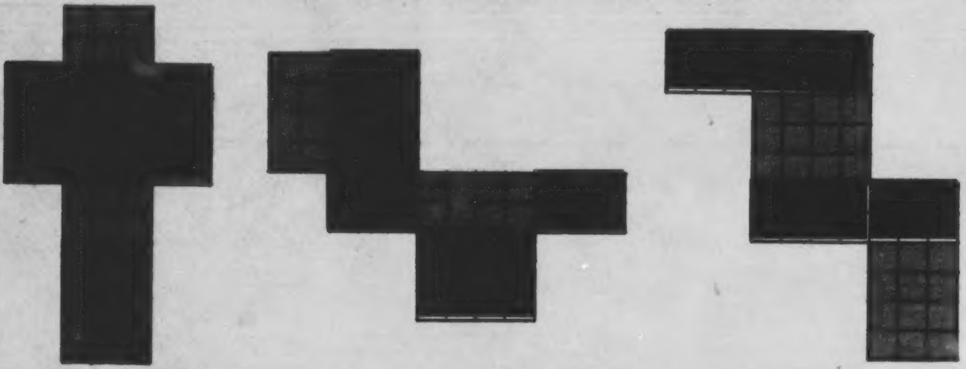
S: Six.

T: Yes. Each face is painted with one color and opposite faces are painted with the same color. So two faces are red, two faces are blue, and two faces are yellow.

G9

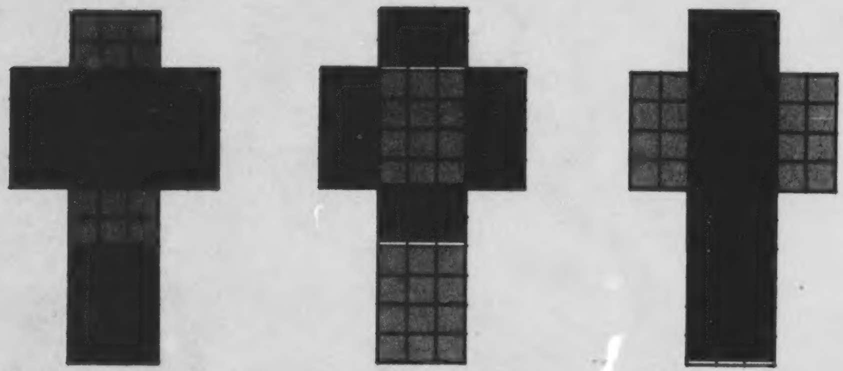
T: I will give you centimeter grid paper and colored pencils. Draw a map of my box and color the six faces. Remember: opposite faces are the same color.

Distribute Worksheet G10, colored pencils, and scissors. There are many possible maps of a 2 cm by 3 cm by 4 cm box. Three are shown here.



Most students will discover a relatively simple map. Circulate to assist those students who have difficulty. If necessary, recall the problem of mapping a cube that appeared in Lesson G1.3.

When most of the class have found at least one map of the box and colored the faces, start to draw students' solutions on the board or, if available, on the grid board. There may be a variety of colorings even for the same map. Some possibilities are shown here.



Ask students to check each solution. A good system for checking is to cut the colored map from a worksheet and to attempt to fold it into a box. Once a box is successfully formed, the opposing faces will be evident and the coloring clearly correct or incorrect.

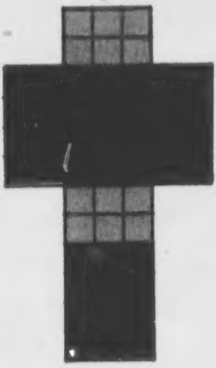
Accept and check solutions until interest wanes. Save as many of the board drawings as possible. Also keep the cutout models. Tape them to the board for reference and possible use later in the lesson.

T: What is the area covered by a map of my box?

Pick one of the maps. Conduct a discussion to calculate the area of the configuration of the six rectangles that comprise a map. Points to explore are:

1) the area of the map is the sum of the areas of the six rectangles;
2) rectangles of the same color have the same dimensions and hence have the same area;
3) the area of the map can be found by calculating the areas of the different colored rectangles and then doubling their sum.

Based on the following map, the discussion might go as follows:



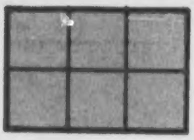
T: This map is made of several pieces.

S: Six pieces, all rectangles.

S: We only have to find the areas of the rectangles.

G 9

Draw a copy of each color rectangle in turn.



What is the area of the red rectangle?

S: 12 cm^2 (3 x 4 = 12).

T: What is the area of the blue rectangle?

S: 8 cm^2 (2 x 4 = 8).

T: And what is the area of the yellow rectangle?

S: 6 cm^2 (2 x 3 = 6).

T: The area of red plus the area of blue plus the area of yellow is ...?

S: 26 cm^2 (12 + 8 + 6 = 26).

S: But there are two red rectangles in the map.

S: And two blue and two yellow rectangles.

S: Two red rectangles together have an area of 24 cm^2.

S: Two blue have an area of 16 cm^2.

S: Two yellow have an area of 12 cm^2.

S: The area of the map is 52 cm^2 (24 + 16 + 12 = 52).

S: That is two times the area of red plus blue plus yellow; 2 × 26 = 52.

Write the area next to the map used.

T: The area of this map is 52 cm^2. That is the total area of all of the faces of my box. The faces together are called the surface of the box. The total of the areas of the faces is called the surface area of the box. What is the surface area of my box?

S: 52 cm^2.

Check the calculation by using another of the maps drawn on the board. The point to make is that the surface area of the box does not depend on the map, so long as the map is correct.

Exercise 2

Distribute individual copies of Worksheet G11.

T: We have found the surface area of my box to be 52 cm^2. What is its volume?

S: 24 cm^3 (2 × 3 × 4 = 24).

T: Can you build any other boxes with the same volume?

S: 6 cm by 2 cm by 2 cm.

S: 1 cm by 1 cm by 24 cm.

T: Each of you think of a box with volume 24 cm^3. Draw a map of your box and calculate its surface area.

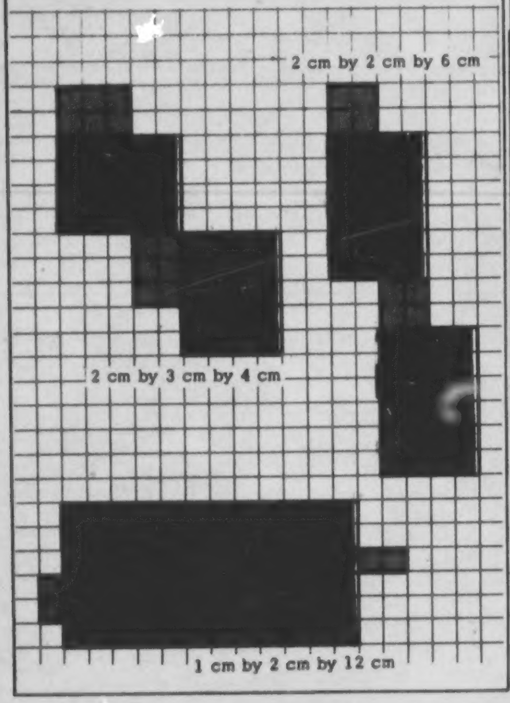
G 9

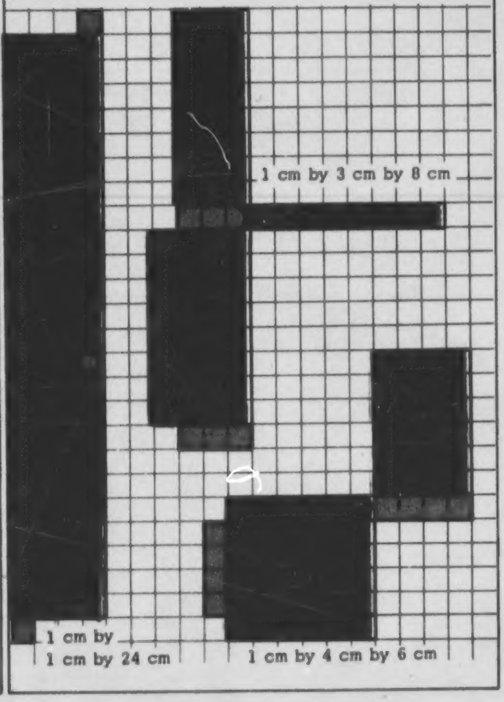
There are six boxes with edges of integer length.

Dimensions	Volume	Surface Area
1 cm by 1 cm by 24 cm	24 cm^3	98 cm^2
1 cm by 2 cm by 12 cm	24 cm^3	76 cm^2
1 cm by 3 cm by 8 cm	24 cm^3	70 cm^2
1 cm by 4 cm by 6 cm	24 cm^3	68 cm^2
2 cm by 2 cm by 6 cm	24 cm^3	56 cm^2
2 cm by 3 cm by 4 cm	24 cm^3	52 cm^2

Encourage those who work quickly to find several boxes with volume 24 cm^3. At the end of the work, lead a discussion comparing the surface area of the boxes. Which is largest? Which is smallest? During the discussion, ask students to draw maps on the board and perform the calculations of the surface area.

Name ________________ G 11


2 cm by 2 cm by 6 cm
2 cm by 3 cm by 4 cm
1 cm by 2 cm by 12 cm


1 cm by 3 cm by 8 cm
1 cm by
1 cm by 24 cm
1 cm by 4 cm by 6 cm

2

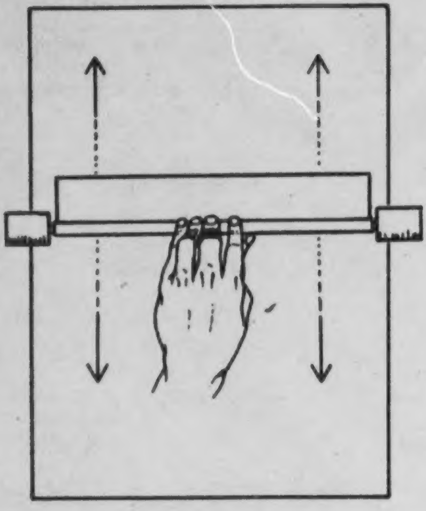
PARALLELISM

Parallelism is at the center of the affine geometry lessons of Parts II and III of CSMP Mathematics for the Intermediate Grades. In the lessons of this chapter, the concepts and constructions related to parallelism are applied to provide a model for an important type of numerical problem. The students are introduced to a fruit market which only offers two commodities, apples and blueberries. To represent purchases in the market, students construct two dimensional lattices with translators. The lattice allows careful and accurate representation of the market transactions. The direct involvement of students in constructing the representation is a compelling support to solving related numerical problems. As the students gain familiarity with the market situation, the context is enhanced by allowing returns and exchanges in the market, which in turn requires an extension of the lattice along the negative axes. After the geometrical model for marketing is well established, prices are introduced and the students attack several instances of the diophantine (linear) equation "ax + by = c". The equation is disguised in questions such as "What transactions (purchases, returns, exchanges) can be made at a value of exactly one dollar?" The lattice provides a means not only for representing solutions, but also for discovering solutions. Moreover, the lattice provides a direct comparison of the solution sets of the various similar equations.

NOTE ON TRANSLATORS

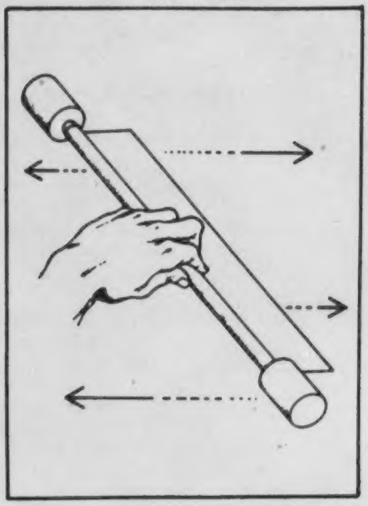
Before using the translator in the classroom, practice using it yourself. Take some time to practice using a student translator as well as the demonstration translator. Occasionally, a construction at the board requires an odd posture. In introducing the class to the translator, mention the sharp corners of the metal blade as a warning against misuse. Provide a few small pieces of masking tape to each student so that paper can be anchored to the desk. This precaution will eliminate frustrating slippage of the paper while a student is drawing. Instruct the class in the proper manipulation of the translator. One should hold the plastic tube at the center and gently push or pull to move it across the paper or desk. The following pictures illustrate proper and improper handling.

Like this

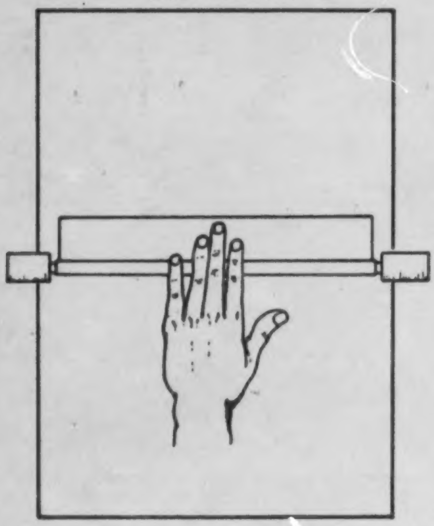


or

Like this

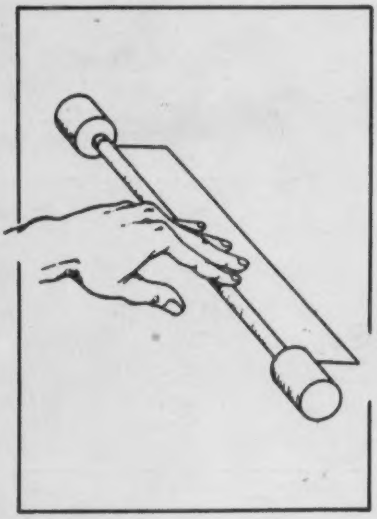


Not like this



and

Not like this



That the blade is not rigidly attached to the plastic tube should cause no problem when a slow steady pressure is used in rolling the translator. The advantage of this means of attachment is that blades may be easily repaired in the event of breakage. For repairs, simply provide yourself with a roll of strapping tape as used in the construction of the translator. However, we anticipate that repairs will only rarely be necessary.

CAPSULE LESSON SUMMARIES

Section 2.1 :	Review the basic parallelogram construction with the translator. Build a two dimensional lattice to represent purchases of apples and blueberries in a fruit market.
Section 2.2 :	Review and extend the use of a two dimensional lattice to represent transactions in a fruit market. Introduce returns and exchanges in the market requiring negative numbers in the lattice.
Section 2.3 :	Study the region of all possible purchases of apples and blueberries in the market. Use parallel projection to interpret points in the lattice. Then find points that represent purchases costing exactly one dollar. These points lie on a straight line.
Section 2.4 :	Given the cost of two purchases, determine the unit prices of apples and blueberries. In a two dimensional lattice, locate points that represent purchases costing exactly sixty cents. Observe that these points lie on a straight line. (Worksheet G12)
Section 2.5 :	Given the cost of two purchases, determine the unit cost of apples and of blueberries. Find even exchanges of apples and blueberries; that is, transactions that can be made without money. (Worksheet G13)
Section 2.6 :	List a variety of transactions at the market and find the cost of each transaction. Locate points on a two dimensional lattice which represent transactions costing 60¢, 100¢, and 0¢ (even exchanges). Observe that these determine three parallel lines. (Worksheet G14)

251

2.1 Purchase Vector Space #1

CAPSULE LESSON SUMMARY

Review the basic parallelogram construction with the translator. Build a two dimensional lattice to represent purchases of apples and blueberries in a fruit market.

MATERIALS

Teacher:	Colored chalk; demonstration translator
Student:	Large sheet of unlined paper; colored pencils; translator

NOTE: It may be useful for you to review the lessons of IG-II and III on Parallelism (see Chapter 2 of the Geometry and Measurement Strand in both Parts II and III). Lesson G2.7 of Part III is especially pertinent.

DESCRIPTION OF LESSON

Depending on the needs of your students, begin the lesson with a review of the basic parallelogram construction as outlined briefly in Exercise 1. Most students will benefit from this review, especially those who have not used the translator recently.

Exercise 1

Initiate a parallelogram construction and, with students' assistance, complete it as outlined below.

The parallelogram construction is easier to perform than to describe. The steps are illustrated below and keyed to this description.

A) First, draw a line segment ①.

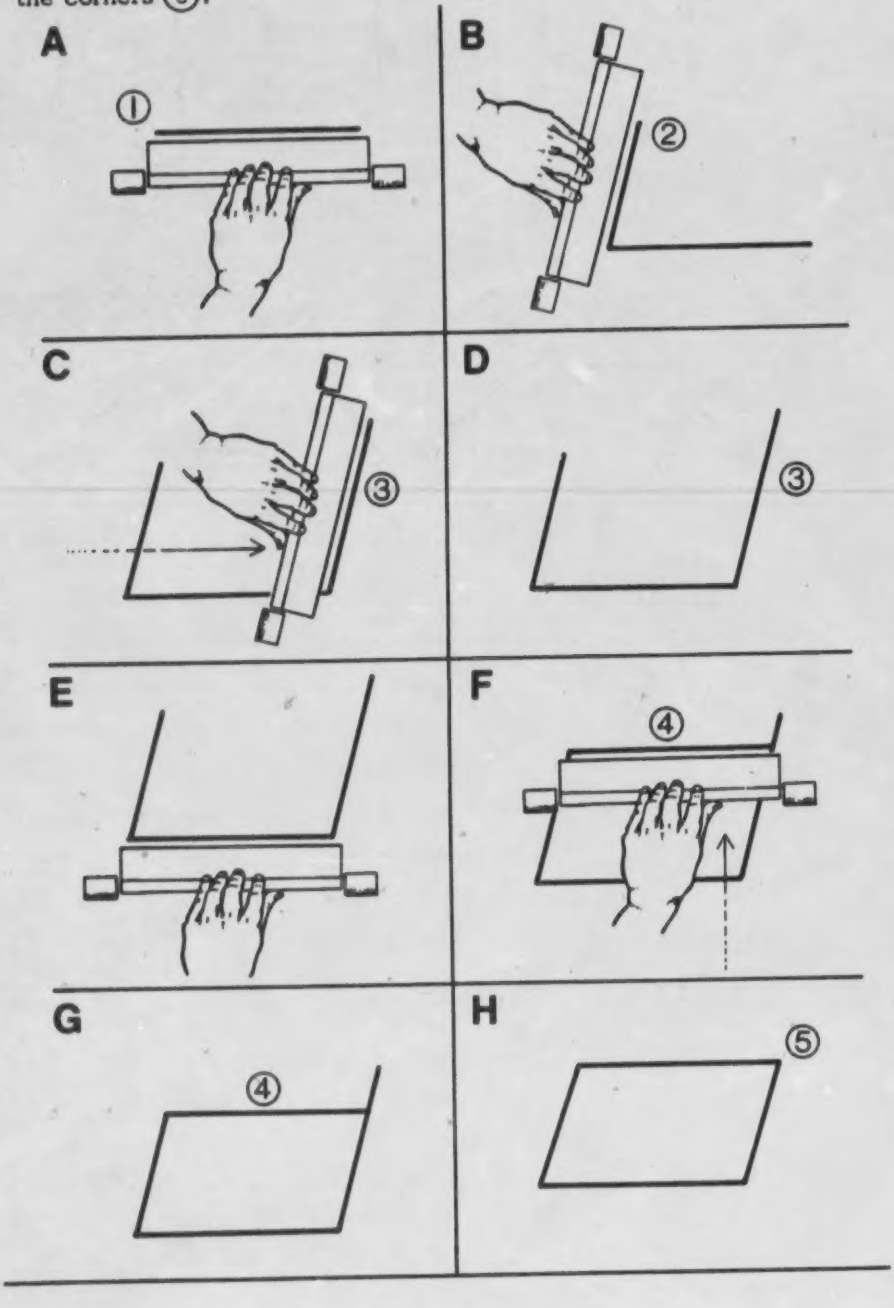
B) Then draw another line segment ② starting at one of the endpoints.

C) and D) Align the translator with the line segment ②, roll to the other end of the first line segment and draw another line segment ③.

E) Align the translator with the first line segment ①.

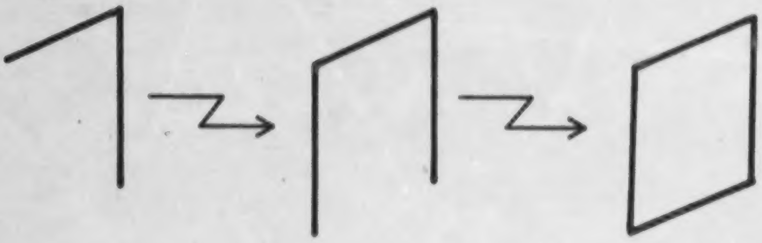
F) and G) Roll the translator to the endpoint of the second line segment ② and draw the last segment ④.

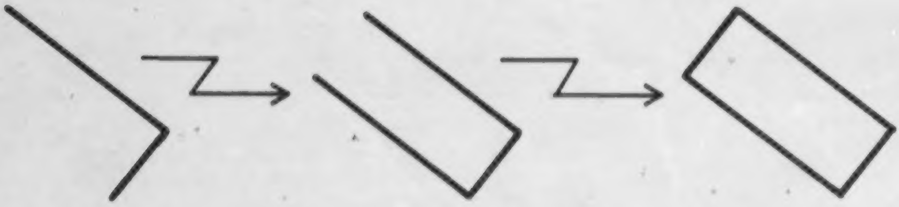
H) Tidy the corners by erasing any pieces of lines which extend beyond the corners ⑤.



G 3

Repeat the construction with one or two other parallelograms of differing orientations. For example,

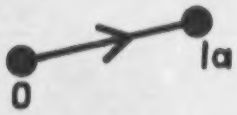




Exercise 2

T: Recently I stopped in a small fruit market. The market was so small that there were only two items for sale: apples and blueberries. I was interested to watch people come and go and make their purchases. I decided to make a record of the purchases. I had a large piece of paper with me, so I drew a picture as a record. First I drew a dot for no purchase at all; I myself had not bought anything. Then I drew a dot for the first purchase which I saw: a woman bought one kilogram of apples. I showed the purchase this way.

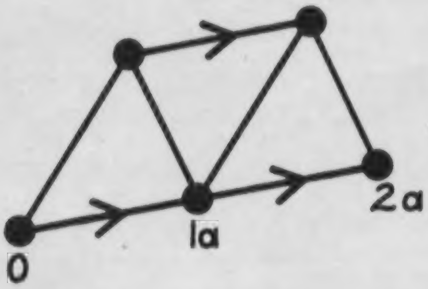
At the appropriate moment in the narrative, draw the dots and label them as shown here. Then draw a red arrow using the translator as a straightedge.



T: The next purchase was of two kilograms of apples. Where should I put a dot for this amount of apples? Let's be accurate.

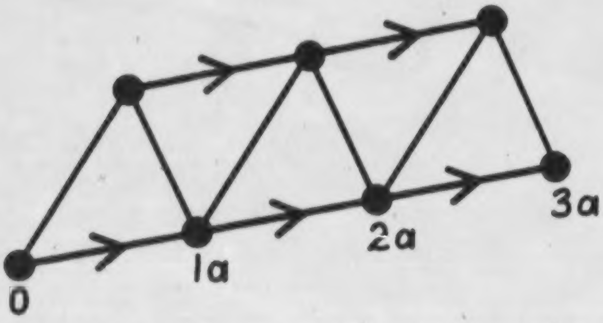
Lead the discussion to a construction of a red couple starting at the dot labeled "1a". (See the similar construction in Lesson G2.7 of Part III.)

Emphasize the use of the translator for accuracy in the construction.



T: Where should I put a dot for 3 kilograms of apples?

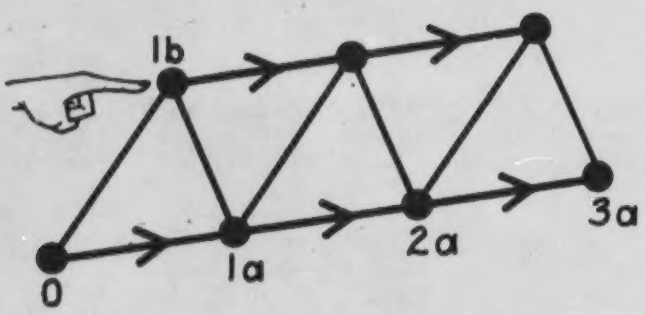
With students' assistance, continue the construction one more step.



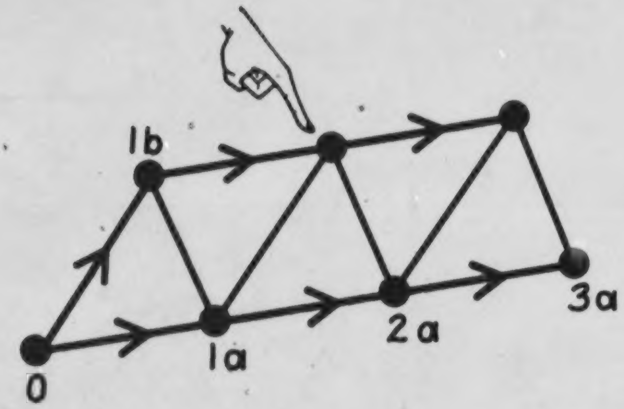
T: Remember that a customer could buy blueberries in the market as well as apples. Our picture should show purchases of blueberries and of apples and of both blueberries and apples. We can use a dot already in the picture to show a purchase of one kilogram of blueberries. Let's use this one.

Point to the dot (see the next illustration) and label it "1b".

G 3



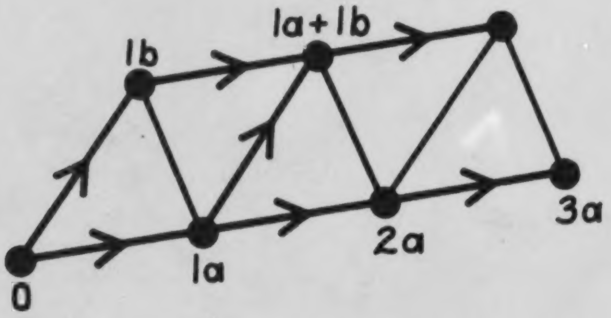
T: Let's use a blue arrow to show a purchase of one kilogram of blueberries. (Replace the segment between the dots labeled "0" and "1b" with a blue arrow.) Now what purchase does this dot represent? (Point to the dot indicated.)



The answer is "1b + 1a". There may be some discussion before general agreement is given. Illustrate the answer by tracing arrows starting at the dot labeled "0" and ending at the dot in question. Follow first the blue arrow (a purchase of 1b) and then the red arrow (a purchase of 1a).

T: There is another way to explain the purchase represented by this dot. We can add another blue arrow to the picture. If we trace first along the red arrow (trace the arrow from 0 to 1a) and then up to this dot, we show first a purchase of one kilogram of apples (trace the red arrow again) and then a purchase of one kilogram of blueberries. Let's use another blue arrow to show the purchase of blueberries.

Replace the segment connecting the dot labeled "1a" and the dot under consideration with a blue arrow. Label the dot "1a + 1b" as you explain again that it represents a purchase of one kilogram of apples and one kilogram of blueberries.

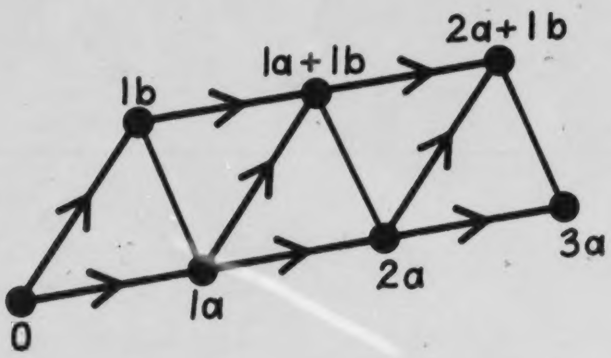


T: How should we label this dot? (Point to the dot to the right of "1a + 1b".)

S: "2a + 1b".

S: There is another blue arrow from 2a to that point.

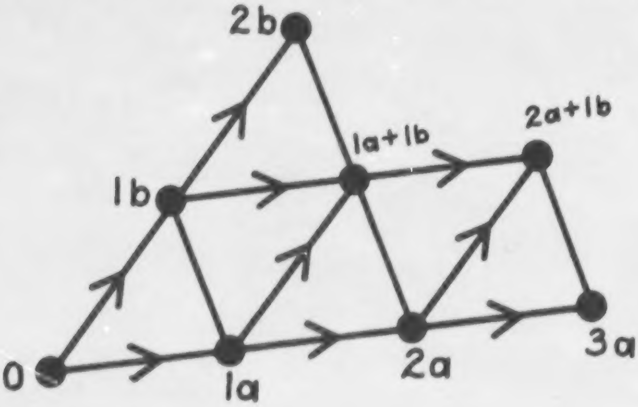
Add a blue arrow starting at 2a. Then justify the labeling of the dot by tracing two red arrows (from 0 to 1a and from 1a to 2a), and one blue arrow (from 2a to the point). Label the point with "2a + 1b" as you explain again that it represents a purchase of two kilograms of apples and one kilogram of blueberries.



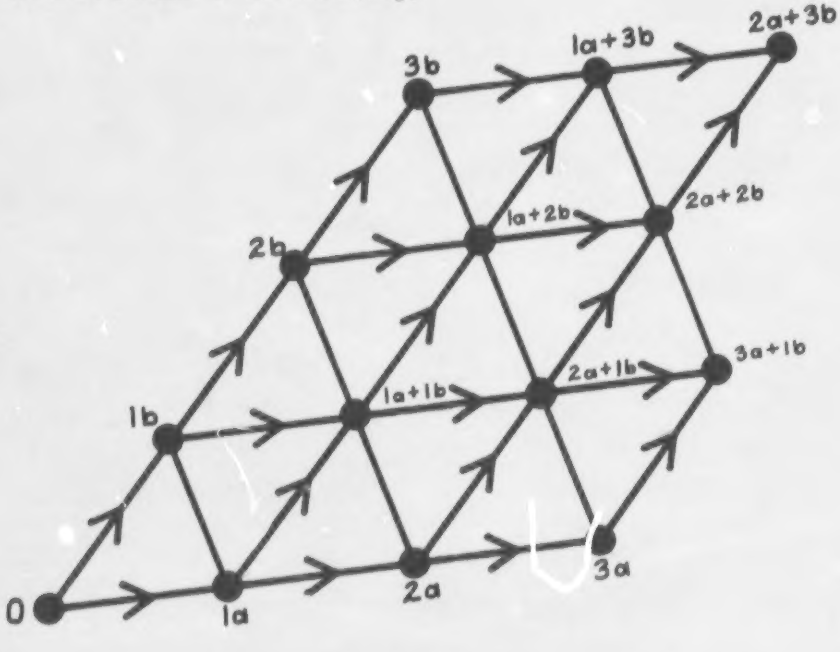
T: Where should we put a dot for a purchase of two kilograms of blueberries?

G 3

Ask students to indicate the location of this dot. There may be a variety of suggestions. Draw them lightly on the board and continue until someone suggests a construction with the translator. The object is to construct a blue couple starting at 1b. This is not an easy construction for many students. Perform the construction at the board with students' assistance.



Continue to expand the lattice by adding more dots and labeling them. For each part of the construction, ask for a student's assistance in using the translator to locate the dots and to label them as they appear. Continue until your drawing extends three or four steps in each direction. Your drawing should be similar to the following.



Distribute large sheets of paper, translators, and colored pencils. Instruct the students to construct market purchase records similar to the one on the board and to label all of their dots. Encourage those students who work quickly and accurately to extend their lattices even further than the one on the board.

2.2 Purchase Vector Space #2

CAPSULE LESSON SUMMARY

Review and extend the use of a two dimensional lattice to represent transactions in a fruit market. Introduce returns and exchanges in the market requiring negative numbers in the lattice.

MATERIALS

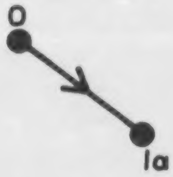
Teacher: Colored chalk; demonstration translator

Student: Large sheet of unlined paper; ruler; colored pencils; translator

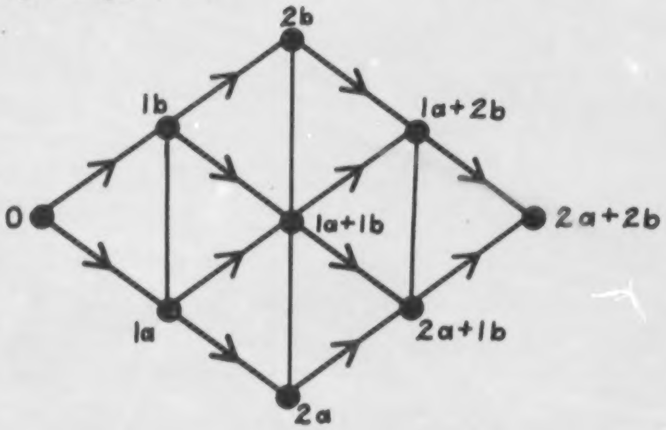
DESCRIPTION OF LESSON

Review the situation (from Lesson G2.1) of keeping a record of market purchases as you begin to draw a picture on the board.

T: Let's think about the fruit market with apples and blueberries again. Here I have a dot for no purchases and here a dot for 1a.



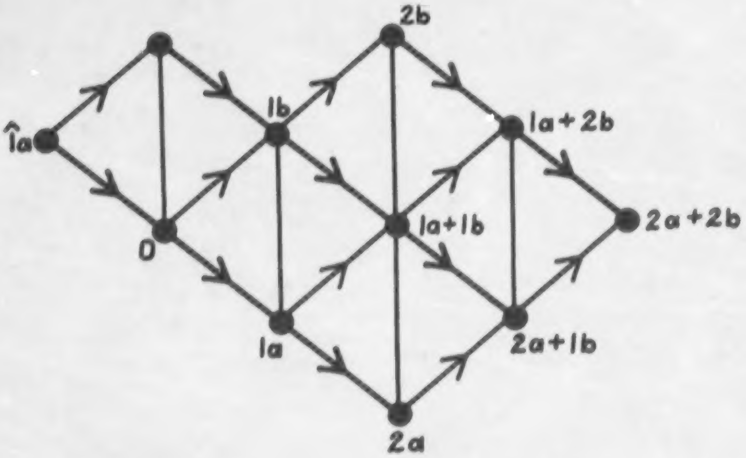
Continue the discussion to expand your drawing. Ask students to locate dots for certain purchases, indicating the proper moves with the translator in the construction, and to label the dots as they appear. Your drawing should include the following configuration.



T: An interesting service of the market is that it accepts returns and exchanges. A person came in and returned one kilogram of apples. How can we show this in the picture?

Lead the discussion to realize that a red couple ending at 0 must be constructed. Then perform the construction with student assistance. There are several parallelograms that might be constructed to locate the starting point; only one is shown in the next illustration. Important: the new red arrow must point in the same direction as the other red arrows. Label the dot "$\hat{1}$a" and, if necessary, remind the class that "$\hat{1}$" is another way of writing "-1" or negative one.

NOTE: Using the "^" notation results in a convenient notation for combinations of purchases and returns.



Continue the discussion to label any remaining unlabeled dots in your drawing. Observe that to move from the dot labeled "0" against an arrow must signify the opposite of a purchase, that is, a return. Each time a dot is labeled, trace (with or against) the arrows in a path starting at 0 and ending at the dot being labeled. For example, the last unlabeled dot in the above drawing is reached by starting at 0, tracing against a red arrow and tracing with a blue arrow. Thus the dot represents a return of one kilogram of apples and a purchase of one kilogram of blueberries. It should be labeled "$\hat{1}$a + 1b".

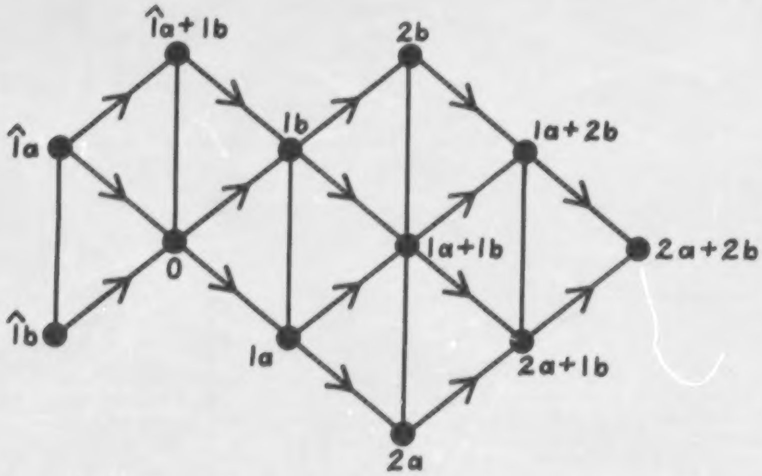
G 4

T: Who can locate a dot to represent a return of one kilogram of blueberries?

Invite a student to direct the moves with the translator to locate the dot correctly. Emphasize the drawing of a blue arrow ending at 0.

T: If we start at 0 and trace against this blue arrow (do so yourself), we show a return of one kilogram of blueberries. How should we label the dot?

S: "$\hat{1}b$".



Distribute large sheets of unlined paper, translators, and colored pencils. Ask students to construct their own pictures of purchases and returns at the market. Instruct them to make a large picture and to locate dots for the following purchases and returns. Write this list on the board.

$1a$

$1b$

$1a + 1b$

$2a + 1b$

$1a + \hat{1}b$

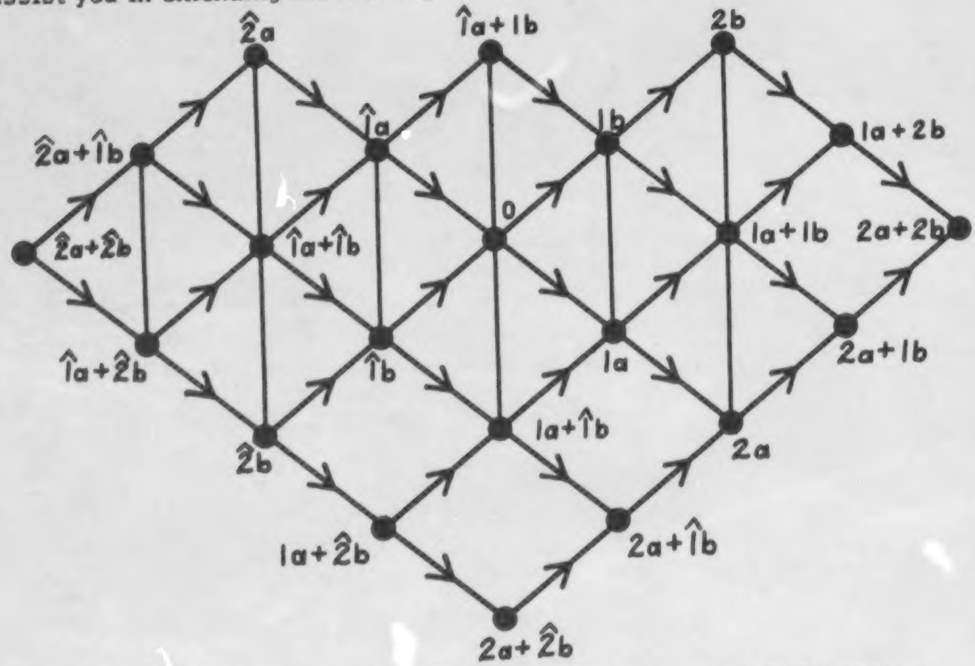
$2a + \hat{1}b$

$\hat{2}a$

$\hat{2}a + \hat{2}b$

$2a + \hat{2}b$

While the students work on their drawings, continue to ask individuals to assist you in extending the drawing on the board.


$\hat{2}a$
$\hat{1}a+1b$
$2b$
$\hat{1}a$
$1b$
$\hat{2}a+\hat{1}b$
$1a+2b$
0
$\hat{2}a+\hat{2}b$
$\hat{1}a+\hat{1}b$
$1a+1b$
$2a+2b$
$\hat{1}a+\hat{2}b$
$\hat{1}b$
$1a$
$2a+1b$
$1a+\hat{1}b$
$\hat{2}b$
$2a$
$1a+\hat{2}b$
$2a+\hat{1}b$
$2a+\hat{2}b$

Encourage students to label all of the dots which appear in their drawings. Those who finish quickly could make even more elaborate drawings with more labeled dots than in the picture on the board.

G5

2.3 Purchase Vector Space #3

CAPSULE LESSON SUMMARY

Study the region of all possible purchases of apples and blueberries in the market. Use parallel projection to interpret points in the lattice. Then find points that represent purchases costing exactly one dollar. These points lie on a straight line.

MATERIALS

Teacher: Colored chalk; demonstration translator
Student: Large sheet of unlined paper; ruler; colored pencils; translator

DESCRIPTION OF LESSON

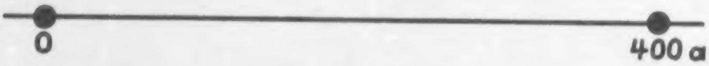
Exercise 1

T: The owner of the fruit market and I became good friends because of my many visits. On one of my visits she said, "Today I have 400 kilograms of apples and 50 kilograms of blueberries to sell. How will you make your picture?" What do you think I drew?

Let the class react and make suggestions before giving any further information. Create a picture on the board as you talk.

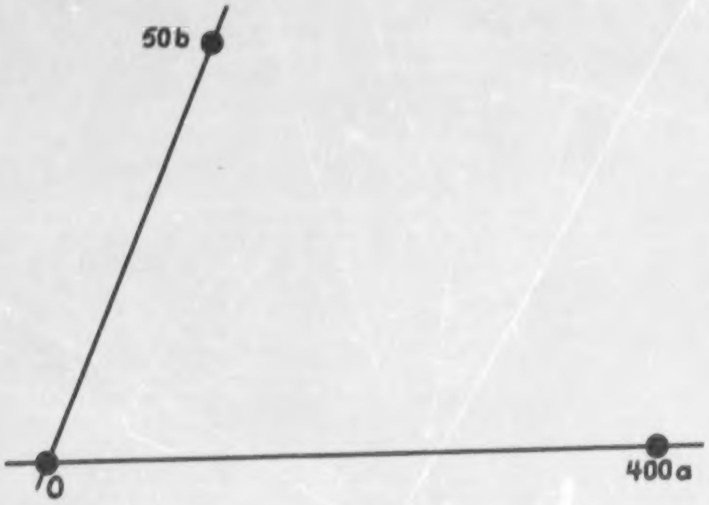
T: First I drew a dot for 0 and a dot for 400 kilograms of apples. Then I drew a line segment connecting them.

NOTE: Do not use colored arrows in constructing the lattice. The emphasis in this lesson is on the relation of various purchases, represented by dots, rather than on their composition as increments of kilograms of apples and blueberries.



T: Then I drew a dot for 50 kilograms of blueberries. Where do you think this dot should be drawn in my picture?

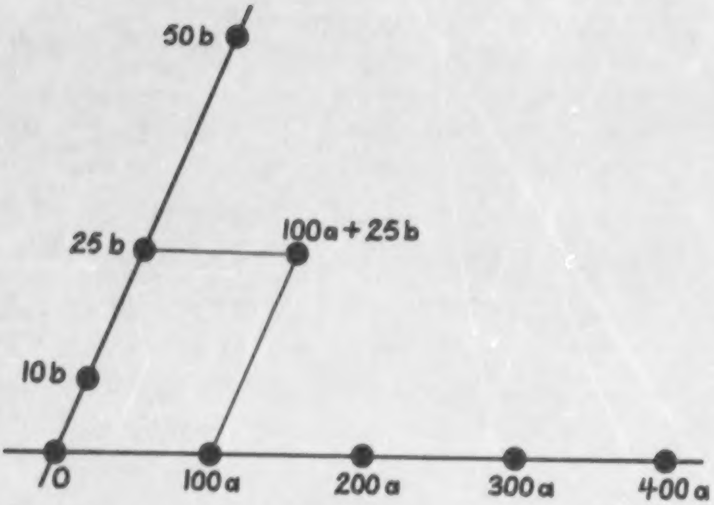
There may be discussion about where to locate a dot for 50b. Some students may suggest placing it on the line already determined for purchases of apples. Of course these points are already determined to represent purchases (or returns) of quantities of apples. If such a point is suggested, ask the class to interpret it as a purchase (or return) of apples. Emphasize that each point has only one interpretation so that, in particular, a single point does not represent both a purchase of 50b and a purchase of some quantity of apples. Eventually students will recognize that another line is needed and will suggest a dot for 50b off the apple line. If necessary, suggest one yourself.



T: Who can show us where to draw the dot for 200a? ... for 100a? ... for 300a? ... for 25b? ... for 10b? ... for 100a + 25b?

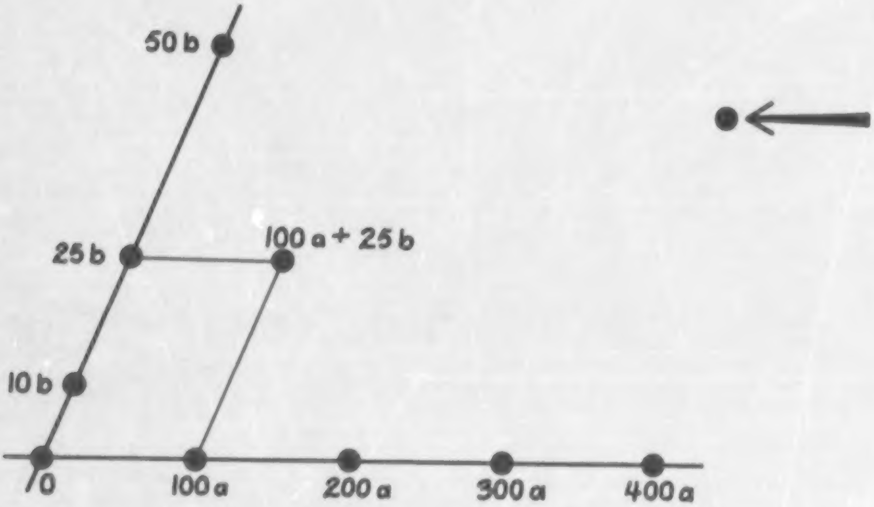
Draw and label each of these dots as they are found. Use a meter stick to place the first five dots accurately; that is, locate 200a at the midpoint between 0 and 400a, and so on. Use the translator to locate the last dot accurately.

G 5

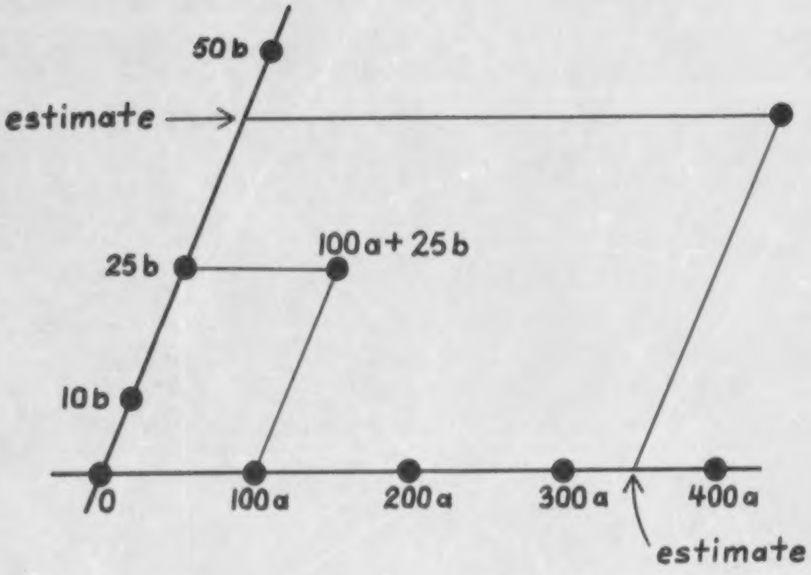

50 b
25 b
100a + 25 b
10 b
0
100 a
200 a
300 a
400 a

Draw a dot in your drawing approximately in the location of 350a + 40b, but do not label it. Such a dot is marked with an arrow in the following illustration.

T: What purchase does this dot represent?

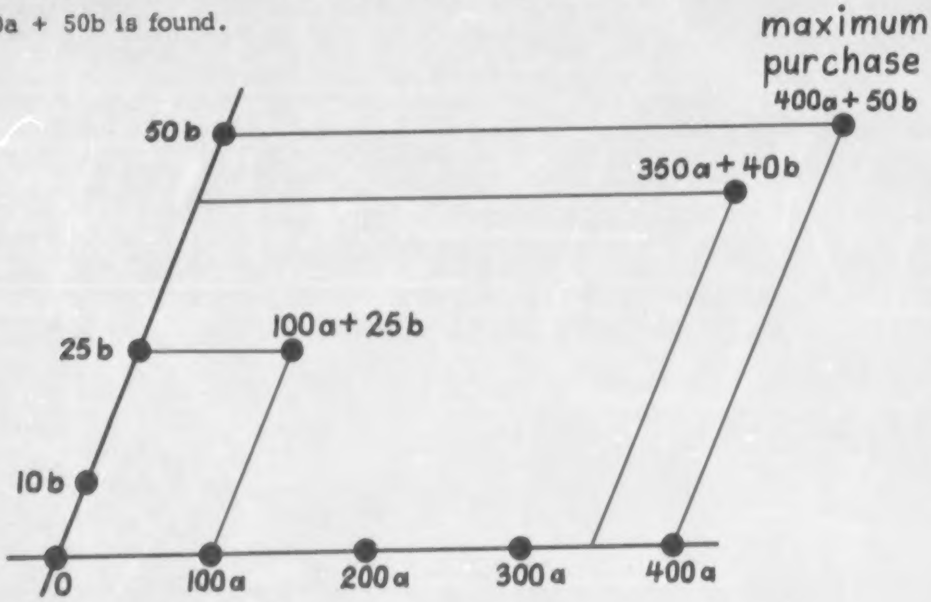

50 b
25 b
100 a + 25 b
10 b
0
100 a
200 a
300 a
400 a

Let the class discuss answers, and lead the discussion to recall that the translator is a good tool to use to read the picture accurately. Perform the projections onto the apple line and the blueberry line and estimate the readings.


50 b
estimate
100a + 25b
25 b
10 b
0
100a
200a
300a
400a
estimate

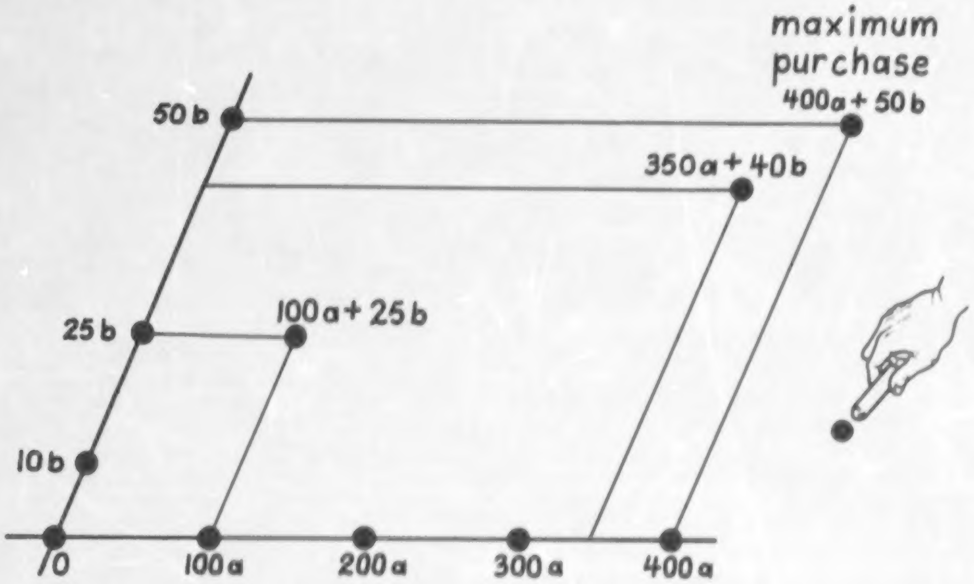
T: Where should we put a dot to represent buying everything in the market? Remember that there are only 400 kilograms of apples and 50 kilograms of blueberries available.

Analyze incorrect answers by projecting with the translator onto the apple line or blueberry line. Continue until the correct (approximate) location of 400a + 50b is found.


maximum purchase
400a + 50b
50 b
350a + 40b
100a + 25b
25 b
10 b
0
100a
200a
300a
400a

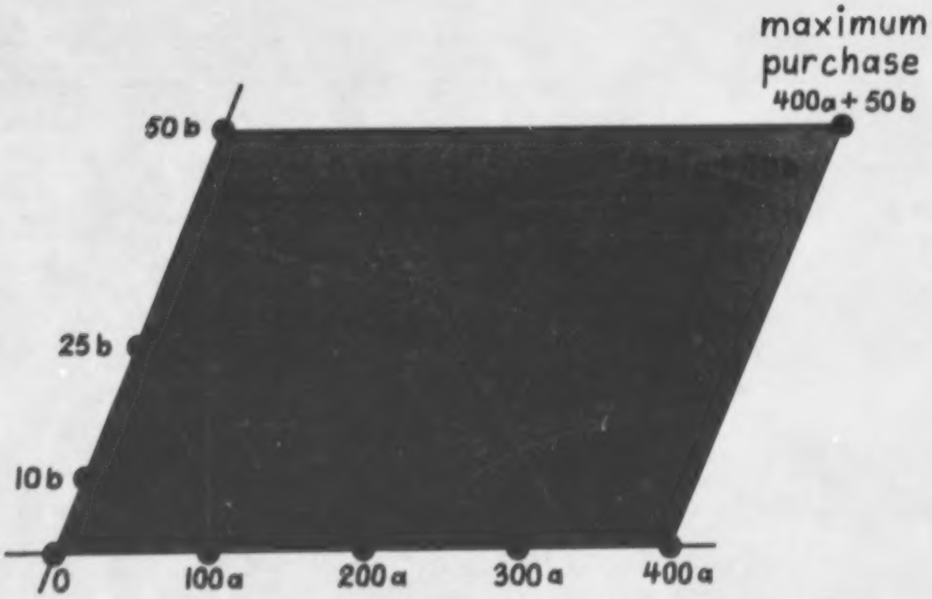
G 5

Draw a dot outside the region bounded by the maximum purchase parallelogram as indicated below. Discuss its interpretation. The class should observe that such a dot is an impossible purchase, since it represents either more apples or more blueberries than the market has to offer.



T: Who can show all possible purchases?

Invite students to respond to this question at the board. Many will be tentative and indicate only a scattering of dots or a portion of the region. Whenever a student suggests some correct possible purchases, ask that the student use colored chalk to mark correct dots or regions. Eventually students' suggested regions will expand. When finished, your drawing will look like the one on the following page.


maximum
purchase
400a + 50b
50 b
25 b
10 b
0
100 a
200 a
300 a
400 a

T: Now we have colored all of the points inside the maximum purchase parallelogram. Should we color the points on the sides of the parallelogram, or just those inside?

S: Color the sides, too.

Exercise 2

T: As in most stores, there is a price for each item at the fruit market. One day, these were the prices.

Write prices on the board for one kilogram of apples and for one kilogram of blueberries.

$\boxed{p(1a) = 20¢}$ $\boxed{p(1b) = 40¢}$

NOTE: "p" denotes "price" and "p(1a)" should be read as "price of one kilogram of apples."

G 5

T: The price of one kilogram of apples was 20¢ and the price of one kilogram of blueberries was 40¢. Suppose that you spent exactly one dollar for fruit. What could you have bought?

As solutions are presented, list them on the board. Each time a solution is suggested, ask the class to check it by calculating the cost of the purchase. The discussion might go as follows.

S: I can buy 3a + 1b.

T: How do we calculate the cost of 3a + 1b?

S: 1a costs 20¢, so 3a cost 60¢ and 1b costs 40¢. The total cost of the purchase is 60¢ + 40¢ or $1.00.

S: I can buy 4a + 0.5b.

S: 4a cost 80¢ and 0.5b costs 20¢, which is half of 40¢, so the cost of 4a + 0.5b is 80¢ + 20¢ or $1.00.

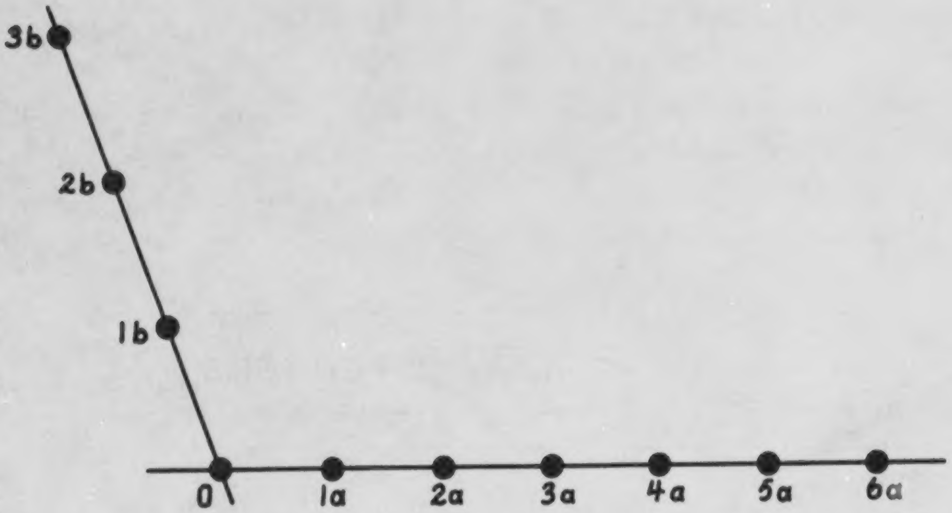
Some of the many other possible solutions are listed here.

5a
3a + 1b
1a + 2b
4a + 0.5b

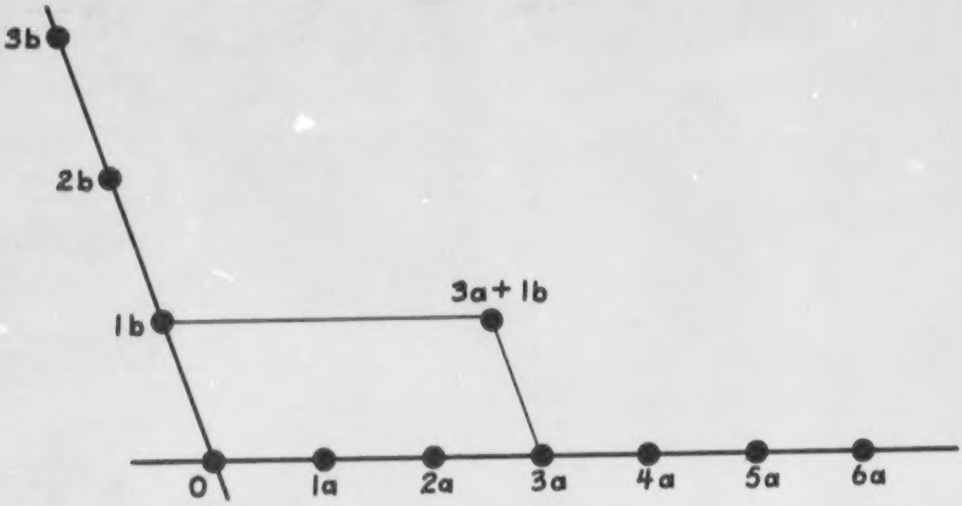
2.5b
2a + 1.5b
0.5a + 2.25b
1.5a + 1.75b

Distribute large sheets of unlined paper, translators, and colored pencils.

Begin to draw a picture on the board as in the next illustration (on the following page) and instruct the students to make similar drawings on their papers. Encourage them to make intervals of uniform length on the "apple" line and the "blueberry" line. Suggest that they use rulers to do this.



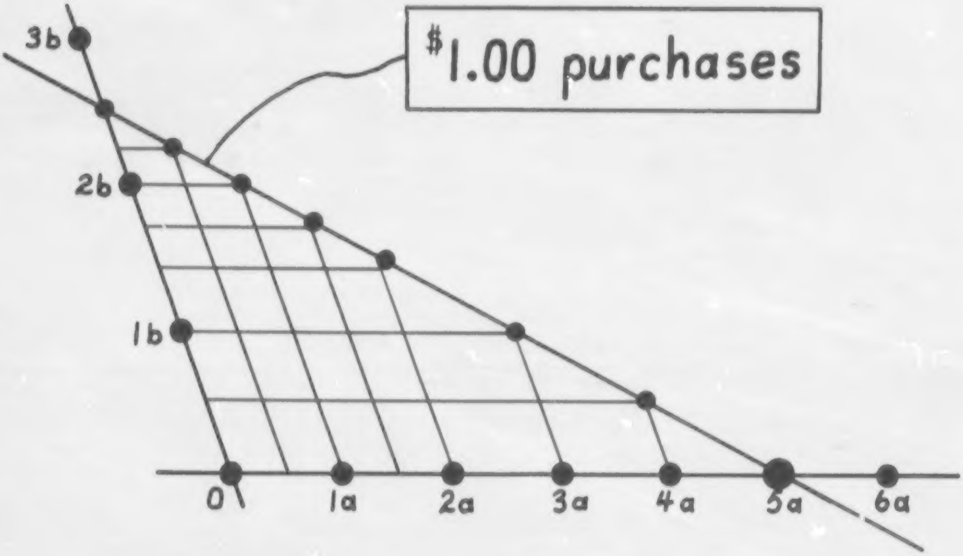
Call on a student to locate (approximately) the dot representing one of the $1.00 purchases, for example, 3a + 1b. Then, with class assistance, use the translator to locate the dot accurately and to color it red. Instruct the class to perform this construction on their own papers as well.



Let the class continue to locate and color red dots representing the $1.00 purchases. Your list on the board provides a number of such purchases for students to represent on their papers.

G 5

As students work and you circulate to assist them with difficulties, occasionally ask a student to locate one of the $1.00 purchases on your board drawing. Constantly encourage the students to work accurately with the translator. Ask those who finish with the list on the board to find other $1.00 purchases, then to construct and label the corresponding dots. At the end, have a short discussion to elicit the observation that the $1.00 purchase dots are all on the same straight line.



2.4 Purchase Vector Space #4

CAPSULE LESSON SUMMARY

Given the cost of two purchases, determine the unit prices of apples and blueberries. In a two dimensional lattice, locate points that represent purchases costing exactly sixty cents. Observe that these points lie on a straight line.

MATERIALS

Teacher: Colored chalk; demonstration translator
Student: Colored pencils; translator; Worksheet G12

DESCRIPTION OF LESSON

NOTE: The first page of the lesson involves determining the unit cost of kilograms of apples and blueberries, given the cost of specified amounts. If this activity would be inappropriate for your class because time consuming or especially difficult for them, then simply start the lesson by announcing the unit costs. The focus of the lesson should be the geometry of the situation and not the arithmetic.

Recall briefly the fruit market where one can buy only apples and blueberries.

T: At the fruit market today, one can buy 2.5 kilograms of apples for 50¢ and can buy 1.5 kilograms of blueberries for 60¢.

Write this information on the board, along with the unit price questions.

$$\boxed{\begin{array}{l} p\,(2.5a) = 50¢ \\ p\,(1a) = ? \end{array}} \qquad \boxed{\begin{array}{l} p\,(1.5b) = 60¢ \\ p\,(1b) = ? \end{array}}$$

Lead a discussion to calculate the prices for one kilogram of apples and one kilogram of blueberries. Try to move through the calculations fairly quickly. The discussion might go as follows.

G8

S: 2.5 kilograms of apples cost 50¢, so twice as much or 5 kilograms of apples will cost \$1.00.

S: If 5 kilograms of apples cost \$1.00, then one kilogram of apples will cost 20¢.

S: 1.5 kilograms of blueberries cost 60¢, so 0.5 kilogram (or $\frac{1}{2}$ kilogram) will cost 20¢. Therefore, 1 kilogram of blueberries costs 40¢.

Add this price information to the list on the board.

$$\boxed{\begin{array}{l} p\,(2.5a) = 50¢ \\ p\,(1a) = 20¢ \end{array}}$$

$$\boxed{\begin{array}{l} p\,(1.5b) = 60¢ \\ p\,(1b) = 40¢ \end{array}}$$

T: What purchases could you make by spending exactly 60¢?

Make a list of student suggestions. Ask the class to check each one by calculating the cost of the purchase. After several solutions have been found, recall to the class that exchanges and returns are allowed and ask for some 60¢ transactions involving exchanges or returns. If the class is slow to think of some other 60¢ transactions, suggest, for example, finding a way to get 5a in exchange for some blueberries and 60¢. (Answer: Return one kilogram of blueberries for a credit of 40¢ to be added to the 60¢; \$1.00 buys five kilograms of apples.) Lead the class to find eight or ten solutions, some of which involve exchanges. Following is a list of possible solutions.

$$3a$$

$$2a + 0.5b$$

$$1a + 1b$$

$$5a + \hat{1}b$$

$$4a + \widehat{0.5}b$$

$$\hat{1}a + 2b$$

$$\hat{2}a + 2.5b$$

$$\hat{3}a + 3b$$

$$0.5a + 1.25b$$

Distribute copies of Worksheet G12, translators, and colored pencils. Instruct students to first label the dots already on the worksheet and then to locate the 60¢ transactions and to color dots for them in blue. Students should locate dots for all of the eight or ten solutions found earlier and then continue to find other 60¢ transactions. Consistently encourage accuracy in the construction with translators.

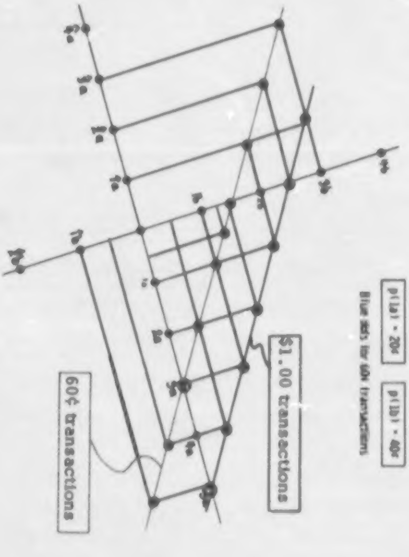
While the individual work is going on, prepare a drawing on the board similar to the one on the worksheet. Occasionally, invite a student to locate one of the 60¢ purchases in your drawing.

Encourage anyone who finishes quickly to locate dots for several $1.00 transactions on the worksheet.

End the lesson with a discussion of the picture directed so as to observe that all of the 60¢ transaction points lie on the same line.

Name ____________________

G 12 *


60¢ transactions
$1.00 transactions
p(la) - 20¢
p(lb) - 40¢
Blue dots for 60¢ transactions

(NOTE: $1.00 transactions are shown for your reference.)

2.5 Purchase Vector Space #5

CAPSULE LESSON SUMMARY

Given the cost of two purchases, determine the unit cost of apples and of blueberries. Find even exchanges of apples and blueberries; that is, transactions that can be made without money.

MATERIALS

Teacher: Colored chalk; demonstration translator
Student: Colored pencils; translator; Worksheet G13

DESCRIPTION OF LESSON

NOTE: The first part of the lesson involves determining the unit cost of kilograms of apples and blueberries, given the cost of specified amounts. If this activity would be inappropriate for your class because time consuming or especially difficult for them, then simply start the lesson by announcing the unit costs. The focus of the lesson should be the geometry of the situation and not the arithmetic.

T: Today the prices at the fruit market are these.

Write the following information and unit price questions on the board.

$$\boxed{\begin{array}{l} p\,(3.5a) = 70¢ \\ p\,(1a) = ? \end{array}} \qquad \boxed{\begin{array}{l} p\,(2.25b) = 90¢ \\ p\,(1b) = ? \end{array}}$$

Lead a discussion similar to that at the beginning of Lesson G2.4 to calculate the price of a kilogram of apples and of a kilogram of blueberries. Move through the calculations fairly quickly. Having determined the prices, add the information to what is on the board.

$$p(3.5a) = 70¢$$
$$p(1a) = 20¢$$

$$p(2.25b) = 90¢$$
$$p(1b) = 40¢$$

T: Let's find even exchanges at the fruit market. No money can be used. Suppose that a woman brings 4a to exchange for blueberries. How many kilograms of blueberries will she receive in an even exchange?

S: 4a are worth 80¢, so she will receive 2b, which are also worth 80¢.

T: Correct; an even exchange. She gives 4a and receives 2b. This even exchange is the transaction $\widehat{4}a + 2b$.

Continue the discussion to find more examples of even exchanges. Ask the class to check the correctness of each suggestion and make a list of the correct suggestions on the board. If students are not forthcoming with examples, suggest part of an even exchange and ask the class to find the other part. Try to get a variety of solutions as in this list of even exchanges.

$\widehat{4}a + 2b$	$4a + \widehat{2}b$	$5a + \widehat{2.5}b$
$\widehat{2}a + 1b$	$2a + \widehat{1}b$	$\widehat{1.5}a + 0.75b$
$\widehat{1}a + 0.5b$	$1a + \widehat{0.5}b$	$2.5a + \widehat{1.25}b$

When you have a list with eight or ten entries, distribute Worksheet G13, translators, and colored pencils. Instruct the students to finish labeling the dots on the worksheet and then to locate and color green dots for even exchanges. Consistently encourage students to make accurate constructions with their translators.

While the individual work is continuing, prepare a drawing on the board similar to the one on the worksheet. Occasionally invite a student to locate a dot for one of the even exchanges in your drawing.

Encourage those who finish quickly to locate several 60¢ purchase dots on their worksheets and to color them with blue dots.

After most students have located dots for all the even exchanges in your list on the board, lead a brief collective discussion, as follows.

T: What can you say about the dots for even exchanges?

S: They all lie on the same line.

Demonstrate by using the blade of your translator as a straightedge. A carefully executed drawing will enhance the likelihood of this observation and of students' conviction in its truth.

T: What about other points on the same line?

Locate a point that is not colored green, but which lies on the even exchange line. Choose one which projects onto easily read quantities of apples and blueberries, for example, $3a + \widehat{1.5}b$. Verify that the point is for an even exchange. Ask students to assist in projecting the chosen point onto the "apple" line and onto the "blueberry" line and in determining the amounts of apples and blueberries. Then calculate the costs. The last step requires that the point be chosen with some care to allow easy and accurate reading of the quantities. Otherwise, if the reading is inaccurate, the even exchange cannot be verified.

Indicate the dot labeled "0".

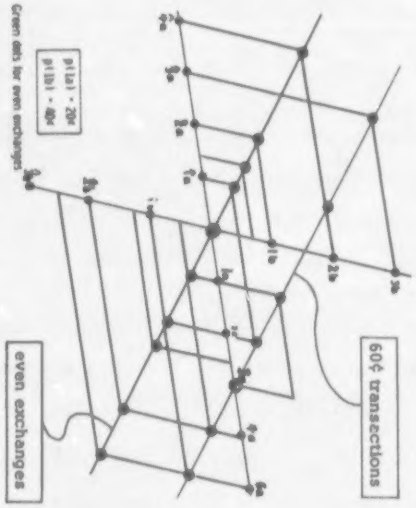
T: What about this dot?

S: No quantity of apples or blueberries given back, and none received; an even exchange.

S: Color it green.

Name ______________________

G 13 *


Green dots for even exchanges
p(a) = 20¢
p(b) = 40¢
1b
2b
3b
even exchanges
60¢ transactions

(NOTE: 60¢ transactions are shown for your reference.)

2.6 Purchase Vector Space #6

CAPSULE LESSON SUMMARY

List a variety of transactions at the market and find the cost of each transaction. Locate points on a two dimensional lattice which represent transactions costing 60¢, 100¢, and 0¢ (even exchanges). Observe that these determine three parallel lines.

MATERIALS

Teacher: Colored chalk; demonstration translator; Geometry Poster #2 for IG IV

Student: Colored pencils; translator; Worksheet G14

DESCRIPTION OF LESSON

T: On my last visit to the fruit market, there were many people making purchases and returns and exchanges. The prices had not changed: 20¢ for one kilogram of apples and 40¢ for one kilogram of blueberries. What are some transactions that could have been made for at most two dollars?

Write the prices on the board and then begin a list of students' suggestions of purchases. When a transaction is suggested, ask the class to calculate its price, and write the price with the transaction as in the following list. Use "¢" notation throughout for consistency in calculations.

$p(1a) = 20\text{¢}$ $\qquad$ $p(1b) = 40\text{¢}$

$2a \rightarrow 40\text{¢}$

$5a \rightarrow 100\text{¢}$

$1a + 1b \rightarrow 60\text{¢}$

$1b \rightarrow 40\text{¢}$

$1a + 2b \rightarrow 100\text{¢}$

$0a \rightarrow 0\text{¢}$

$5a + \hat{1}b \rightarrow 60\text{¢}$

$\hat{2}a + 1b \rightarrow 0\text{¢}$

$4b \rightarrow 160\text{¢}$

$5a + 2b \rightarrow 180\text{¢}$

$2.5b \rightarrow 100\text{¢}$

$0.5a \rightarrow 10\text{¢}$

$\hat{1}a + 3b \rightarrow 100\text{¢}$

$1.5b \rightarrow 60\text{¢}$

$1.5a + 1b \rightarrow 70\text{¢}$

G 11

Part of the discussion might be as follows.

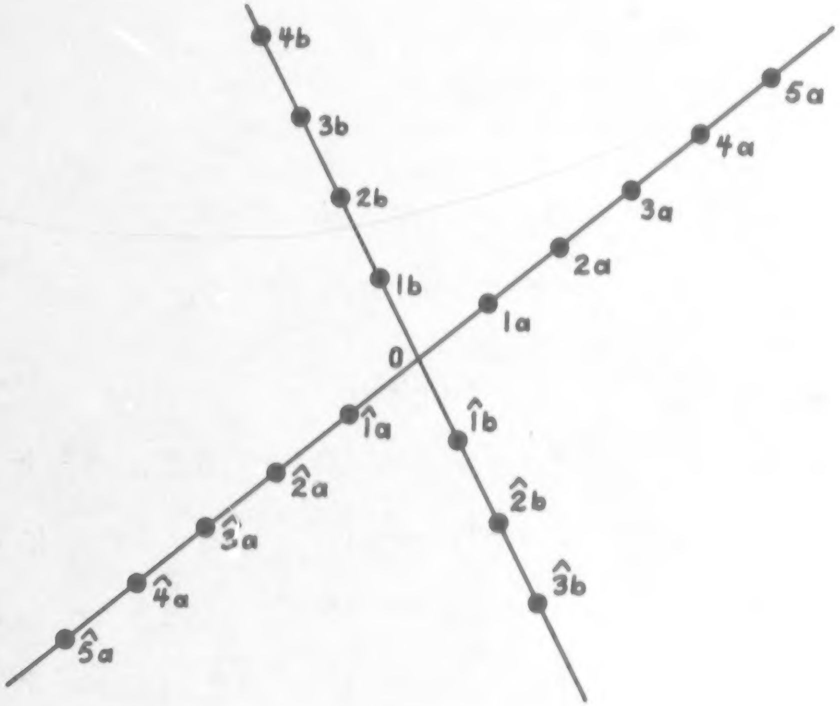
S: I could buy 1a + 1b. A kilogram of apples costs 20¢ and a kilogram of blueberries costs 40¢, so 1a + 1b will cost 60¢.

S: I could buy 2.5b for one dollar.

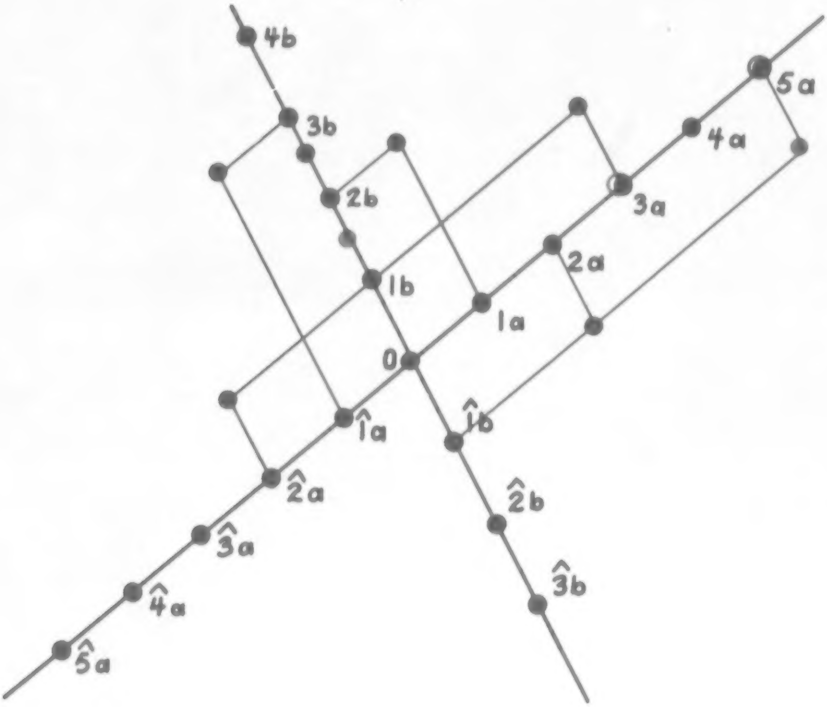
S: $\hat{1}$a + 3b is an exchange that can be made with one dollar, since 3b costs 120¢ while the returned kilogram of apples gives a credit of 20¢.

Continue to make your list on the board until there are ten or twelve entries. The focus of the lesson will be on transactions with prices of 60¢, of 100¢ and of 0¢ (the even exchanges). If none of the suggested transactions have these prices, ask for some specifically. Elicit at least two or three of each.

Draw the picture from Worksheet G14 on the board.



Invite students to assist you in using the translator to locate points to represent transactions with prices of 0¢, 60¢, and 100¢. You choose the price each time. When a purchase that appears in the list on the board is suggested, circle it with the same color as used for the corresponding dot in the drawing. Use red for 100¢, blue for 60¢, and green for 0¢ (the even exchanges).



When all of the transactions from your list with these three prices are located in the board drawing, distribute Worksheet G14. Instruct the students to represent the transactions on the worksheet using three colors as you have done on the board. Students should find even more transactions with the three prices and then locate the corresponding dots in their drawings.

Monitor students' work and use of the translator for accurate placement of dots. As students find transactions other than those in your table, ask them to place them in your board drawing.

G 11

At the end of the lesson lead a brief discussion of the drawings as follows.

T: What do you notice about the red dots for 100¢ purchases?

S: They are all on the same line.

S: The blue dots are all on a line too.

S: So are the green dots.

Display Geometry Poster #2 for IG IV.

S: The red, the blue, and the green lines are parallel!

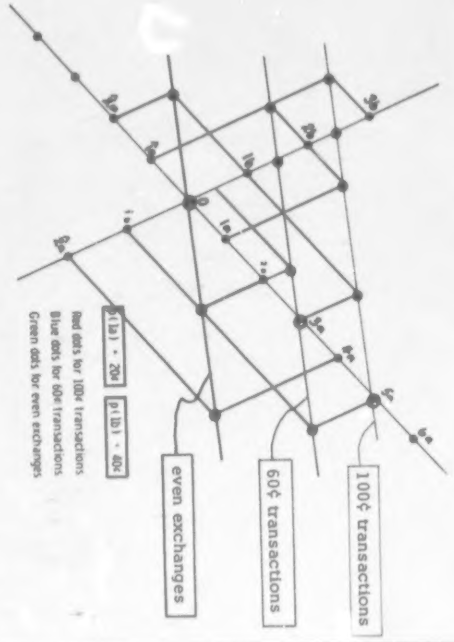
T: How can we check that?

S: Use the translator.

S (all together): PARALLEL, PARALLEL, PARALLEL; ROLLLLL!

Name ______________________

G 14 *


0
p(1a) = 20¢
p(1b) = 40¢
Red dots for 100¢ transactions
Blue dots for 60¢ transactions
Green dots for even exchanges
even exchanges
60¢ transactions
100¢ transactions

PROBABILITY AND STATISTICS

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades

Part IV

Probability and Statistics

CEMREL, Inc.

TABLE OF CONTENTS

SUGGESTED CHRONOLOGICAL ORDER OF P - LESSONS v

INTRODUCTION vii

CHAPTER 1 : RANDOM NUMBER STORIES 1
1.1 Random Number Stories #1 3
Worksheets P01, P02 13
1.2 Random Number Stories #2 14
1.3 ; Random Number Stories #3 22
Worksheets P03, P04, P05, P06 34

CHAPTER 2 : PROBABILITY GAMES 35
2.1 Probability Games #1 37
2.2 Probability Games #2 55
2.3 Probability Games #3 72
Worksheet P08 90

SUGGESTED CHRONOLOGICAL ORDER OF P - LESSONS

Lesson One P1.1 - Random Number Stories #1 3
Lesson Two P1.2 - Random Number Stories #2 14
Lesson Three P2.1 - Probability Games #1 37
Lesson Four P2.2 - Probability Games #2 55
Lesson Five P2.3 - Probability Games #3 72
Lesson Six P1.3 - Random Number Stories #3 22

PROBABILITY AND STATISTICS

INTRODUCTION

There are several reasons why people study mathematics. It is an interesting and challenging intellectual exercise when studied for its own sake. But it is also a supremely useful topic of study in that it has applications outside itself to other disciplines, as well as more generally to understanding the world around us. Some areas of mathematics are more easily applied than others, and the areas which particularly concern us in this strand are towards the easy end of the spectrum, both because of the small amount of background mathematical knowledge required before meaningful application can be made, and because it is usually fairly easy to see the link between the mathematical model and the "real" situation being modeled. For these reasons, it would be difficult to overstate the importance in today's world of probability and statistics and their applications to problems and situations within and outside the discipline of mathematics.

The lessons of this strand deal with systematic methods of analyzing games of chance and of finding numerical probabilities for outcomes or events. Situations which involve us in finding probabilities of certain events give rise to combinational problems and give support to our work with fractions in the World of Numbers strand. Furthermore, the experiments described in the lessons of Chapter 1 exploit relations on the whole numbers and number stories involving these relations, so there is a further integration with the N-lessons and with the L-lessons. The lessons of Chapter 2 makes use of a geometric (measure) method for solving problems involving probability games.

1

RANDOM NUMBER STORIES

The lessons of this chapter are concerned primarily with combinatorics problems. Initially, an experiment is presented in which a random selection is made from five (or three) boxes containing mathematical symbols (numbers, operations, and relations) and a number story is formed. Several trials of the experiment are performed to observe some possibilities of outcomes and to initiate discussions of predicting the truth or falsity of the resulting number story. The overriding problem of determining the probability of getting a true statement is posed. Then experimentation is abandoned and attention focuses on finding a counting technique to determine the number of possible true statements and the total number of possible statements which can be formed. The main conbinatorial principle involved in these situations is the "multiplication rule", and this principle is illustrated by using operation and relation tables.

CAPSULE LESSON SUMMARIES

Section 1.1 : Examine number stories that can be formed by selecting one symbol from each of five boxes where the first box contains three numbers, the second box contains three operations, the third box contains four numbers, the fourth box contains two relations, and the fifth box contains five numbers. Discuss the probability of getting a true number story when a random selection is made. Begin a search for all the possible true number stories.
(Worksheet P01)

Section 1.2 : Review the five boxes experiment introduced in Section 1.1. Extend the exploration and discussion to determine how many possible true number stories there are in this situation and how many possible number stories altogether (true or false) there are. Note that the probability of getting a true number story is about $\frac{1}{3}$ or 1 chance out of 3.
(Worksheet P02)

Section 1.3: Examine number stories that can be formed by selecting one symbol from each of three boxes where the first box contains five numbers, the second box contains four relations, and the third box contains six numbers. Discuss the probability of getting a true number story when random selection is made. (Optional) Pose a problem in a similar situation of determining which numbers to put into the boxes so as to have the largest (smallest) possible probability of getting a true number story. (Worksheets P03, P04, P05, P06)

1.1 Random Number Stories #1

CAPSULE LESSON SUMMARY

Examine number stories that can be formed by selecting one symbol from each of five boxes where the first box contains three numbers, the second box contains three operations, the third box contains four numbers, the fourth box contains two relations, and the fifth box contains five numbers. Discuss the probability of getting a true number story when a random selection is made. Begin a search for all the possible true number stories.

MATERIALS

Teacher: Five boxes; pieces of paper (about 10 cm by 10 cm)
Student: Paper; pencil; Worksheet P01

NOTE: Prior to presentation of this lesson, prepare five boxes with labels "I", "II", "III", "IV", and "V". Into the boxes, put pieces of paper marked as indicated below. Fold the pieces of paper so as to hide what is written on them.

I: 5 7 20
II: + − ×
III: 4 6 8 9
IV: = <
V: 11 28 40 100 200

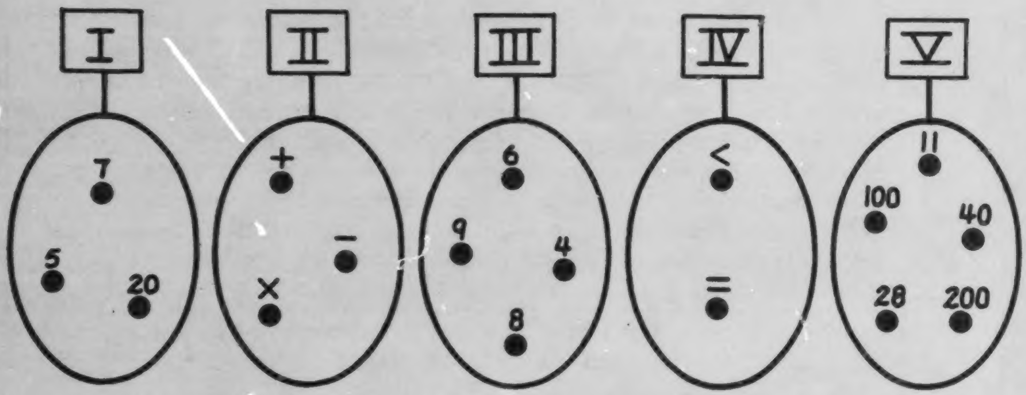
DESCRIPTION OF LESSON

Due to the randomness of the activities in this lesson, you should be prepared to adapt the discussion to the results in your classroom.

T: In these five boxes there are some little pieces of paper, each with a symbol written on it and folded so that the symbol is hidden. The picture I draw will show what is in the boxes.

P 1

As you draw this string picture on the board, read the symbols and explain what is in each box (string).



NOTE: Be sure your students understand that the symbol "< " is read "is less than" or "is smaller than". If necessary, provide a few examples.

T: Now, I'll shake each of the boxes and one of you will select, without looking, one piece of paper from each box.

Choose a student to perform this experiment. Ask the student to read what is written on each piece of paper immediately after it is selected and then return it to the box. Progressively record the results on the board as a number story. For example,

$$20 + 6 = 40$$

T: The result is a number story. Is it true or false?

S: False.

Indicate this information next to the number story on the board.

$$20 + 6 = 40 \quad F$$

T: Let's make another trial and get another number story. First I shake the boxes, then one of you select one slip of paper from each box.

Choose a second student to perform the experiment. Proceed just as before, except stop the student after the fourth box. You might have this expression on the board.

$$20 - 9 =$$

T: Wait a moment. Before you select from the fifth box, can we predict whether the number story we'll get will be true or false?

S: It could be true, if 11 is picked. Otherwise, it will be false.

T: Which is more likely?

S: The chances are better that it will be false.

S: Only 1 chance out of 5 that it will be true.

Instruct the student to now select from the fifth box and you record the result on the board. For example,

$$20 - 9 = 11$$

T: Is this number story true or false?

S: True.

Indicate this information next to the number story.

$$20 - 9 = 11 \quad \text{T}$$

P 1

Call on a couple more students to perform trials of this experiment using the same procedure. Then reverse the order in which the selection from the boxes is made. Ask a student to begin the selection from the fifth box, next from the fourth box, and then to stop a moment. Write the results thus far on the board; for example,

$$= 28$$

T: Can you predict whether the number story we'll get will be true or false?

S: It could be true, because we could get "20 + 8 = 28".

S: Also, we could get "7 x 4 = 28".

S: All the other number stories we could get are false.

Allow students to discuss the chances, but do not insist on a full discussion at this point. Then ask the student to complete the selection while you record the results.

Repeat this reverse order selection. From the fifth and fourth boxes, the selection might be:

$$< 100$$

T: Can you already predict whether the number story we'll get will be true or false?

S: I think it will be true.

S: It could be false, for example, if we get "20 x 6 < 100" or "20 x 8 < 100" or "20 x 9 < 100".

T: Is true or false more likely?

S: True.

Let the student complete the selection while you record the results.

After six trials, you should have a list of six number stories such as these.

$20 + 6 = 40$ F
$20 - 9 = 11$ T
$5 - 6 < 200$ T
$7 - 9 < 28$ T
$5 + 6 = 28$ F
$5 - 4 < 100$ T

T: How many of these number stories are true? How many are false?

S: Four are true and two are false.

T: If we continue making trials, do you think we will get four true number stories for two false ones all the time?

S: We don't know.

S: There is luck in this game.

T: Do you know the probability of getting a true number story?

Let the class discuss this question. Direct the discussion to begin looking at how many true and how many false number stories can be formed.

T: In our six number stories, all those using "<" are true. Are there any false number stories using "<"? If you find some, write them on your paper.

P 1

After a few minutes of independent work, accept examples from the students and record some of them on the board. A few possible examples are presented here.

$5 + 6 < 11$ F
$7 \times 8 < 40$ F
$20 + 9 < 28$ F
$20 \times 9 < 100$ F

The class will observe that there are many false number stories using "< ".

T: Are there more true number stories using "< " or more using " = "?

S: I think there are more using "< ", because there are only a few true number stories using " = ".

T: Try to find all the true number stories using " = ". Write them on your paper.

After many students have found them all, call on students to read these number stories as you record them on the board.

$20 - 9 = 11$ T
$20 + 8 = 28$ T
$7 \times 4 = 28$ T
$7 + 4 = 11$ T
$5 \times 8 = 40$ T
$5 + 6 = 11$ T

T: It wasn't so difficult to find that there are six true number stories using "=". Can you also find some true number stories using "<"? Write some on your paper.

After a few minutes, let students announce examples they have found as you record them on the board. Accept several examples and then ask:

T: We could go on recording these true number stories using "<". If we find them all, will we be able to answer the question of what is the probability of getting a true number story?

S: There are too many true number stories using "<".

S: We would also need to know how many false number stories there are. We need to see whether there are more true or more false number stories.

NOTE: To answer the question of what is the probability of getting a true number story, we first answer two questions:

(i) How many number stories can be formed?
(ii) How many of these are true?

Even if some students pose the first question (and perhaps answer it), we suggest you continue to look for true number stories. This will allow students to progressively become familar with the product rule for counting the possibilities. For most students, the original question will not be answered until Lesson P 1.2. In this lesson we are only setting the stage.

T: How many true number stories did we find using "="?

S: Six.

P 1

Write this information on the board.

True for = ⑥

T: How can we find how many true number stories there are using "< "?

Let students give suggestions. Encourage them to organize the search by finding, for example, all those ending with "< 11 " or "< 200 ". Direct the observations by asking:

T: What is the largest number you can get by selecting from the first three boxes?

S: 180, because 20 x 9 = 180.

T: How can you be sure?

S: 20 is the largest number in the first box, and x gives larger numbers than + or – , and 9 is the largest number in the third box.

T: Are there any false number stories which end with "< 200 "?

S: No.

T: How many true number stories end with "< 200 "?

S: 36.

T: How did you find 36?

S: 3 x 3 x 4 = 36.

S: There are 3 numbers in the first box. For each of these numbers, there are 3 operation symbols in the second box to put with it. That makes 9 (3 × 3) possibilities from the first two boxes. Then for each of these 9 possibilities, there are 4 numbers in the third box to put with it. That makes 36 (9 × 4) possibilities. Add "< 200" to the end of these 36 possibilities and you get 36 true number stories.

Do not insist on this clear explanation of the product rule for counting the possibilities. In fact, do not be concerned if the class does not come up with the correct number at this point. There will be other opportunities to explore this same counting situation.

T: Are there other true number stories using "<"?

S: Yes.

T: Let's look at all the possibilities of number names we can get using the first three boxes.

Make a table on the board for one of three operations in the second box, perhaps one chosen by a student. Explain to the class that the numbers in the first box appear vertically under "×" and the numbers in the third string appear horizontally to the right of "×".

×	4	6	8	9
5				
7				
20				

Let a student choose a number from the first box and a number from the third box and calculate the product; for example 7 × 8 = 56.

T: Where should we write "56" in this table?

P 1

Invite a student to fill in the correct space in the table.

×	4	6	8	9
5				
7			56	
20				

T: How many entries are there in this table?

S: 12.

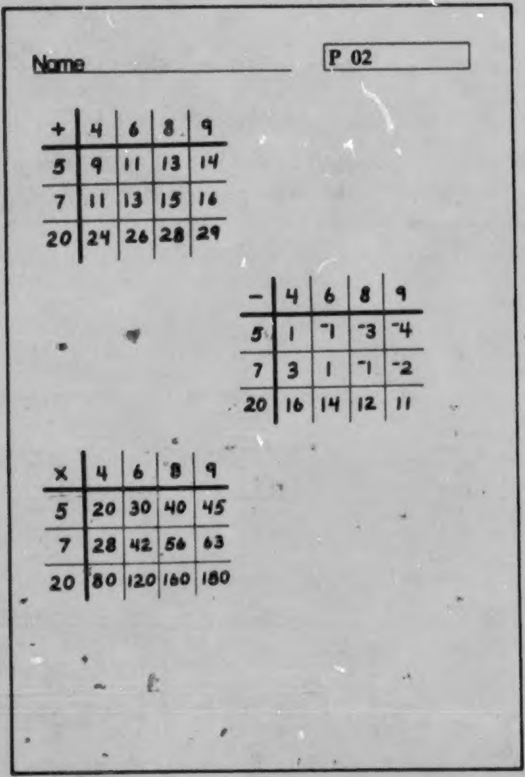
T: These entries are for all the possibilities we get using the first box, "x", and the second box. We can make a similar table for + and for -. How many possibilities in all?

S: 12 + 12 + 12 or 36.

Distribute Worksheet P01 and ask the students to fill in the tables.

Save the boxes and pieces of paper for use in Lesson P 1.2.

Worksheet P02 is an answer key for Worksheet P01.



Name

P 02

+	4	6	8	9
5	9	11	13	14
7	11	13	15	16
20	24	26	28	29

−	4	6	8	9
5	1	$^-1$	$^-3$	$^-4$
7	3	1	$^-1$	$^-2$
20	16	14	12	11

×	4	6	8	9
5	20	30	40	45
7	28	42	56	63
20	80	120	160	180

P 2

1.2 Random Number Stories #2

CAPSULE LESSON SUMMARY

Review the five boxes experiment introduced in Lesson P 1.1. Extend the exploration and discussion to determine how many possible true number stories there are in this situation and how many possible number stories altogether (true or false) there are. Note that the probability of getting a true number story is about $\frac{1}{3}$ or 1 chance out of 3.

MATERIALS

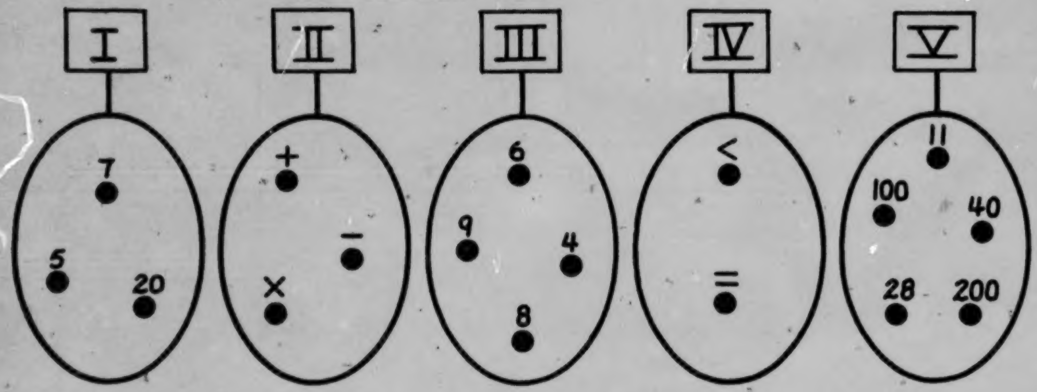
Teacher: Five boxes; specially marked pieces of paper; colored chalk

Student: Colored pencils; Worksheet P02

NOTE: See the note on materials for Lesson L 1.1 for a description of the five boxes and specially marked pieces of paper. Use the same content for the boxes as in that lesson.

DESCRIPTION OF LESSON

Draw a string picture on the board showing the contents of the five boxes.



T: Do you remember the five boxes from last week's lesson? My picture shows what is in each box. What did we do with these boxes?

S: We selected a piece of paper from each box and made number stories.

S: We noted if the number stories were true or false.

T: Can you give some examples of true number stories we could get and some examples of false number stories we could get?

Accept suggestions from the students, each time asking the student to announce whether the number story is true or false. Record examples on the board until you have three or four of each kind. Then begin to review the problem from last week's lesson.

T: In last week's lesson, I posed a problem about this experiment. What was our problem?

S: How many true number stories are there?

T: Why did we want to know how many true nu[illegible]r stories?

S: To help us find the probability of getting a true number story.

T: What information did we find last week?

S: There are six true number stories using "=".

S: All the number stories ending with "< 200" are true.

S: We also need to find how many number stories there are, both true and false.

Let the class recall what they can from last week's lesson. Then distribute Worksheet P02 with the completed tables.

T: These tables show all the numbers we can form when we select one paper from each of the first three boxes.

Discuss briefly how the tables were made, and copy the tables on the board.

P 2

T: We found six true number stories using "=".

T: Let's use these tables to recall those number stories.

Instruct the students to circle in red a result in the table which contributes to a true number story and to read the corresponding number story. Record the responses on the board.

+	4	6	8	9
5	9	⑪	13	14
7	⑪	13	15	16
20	24	26	㉘	29

$5 + 6 = 11$
$7 + 4 = 11$
$20 + 8 = 28$

−	4	6	8	9
5	1	$^{-}1$	$^{-}3$	$^{-}4$
7	3	1	$^{-}1$	$^{-}2$
20	16	14	12	⑪

$20 - 9 = 11$

×	4	6	8	9
5	20	30	㊵	45
7	㉘	42	56	63
20	80	120	160	180

$5 \times 8 = 40$
$7 \times 4 = 28$

Write this information about true number stories using "=" to one side of the board.

true for = ⑥

T: We also found that all the number stories ending with "< 200" were true. How many true number stories are there that end with "< 200"?

S: 36.

T: Can you use these tables to explain your answer?

S: All the numbers in these tables are smaller than 200, and there are 36 entries -- 12 in each table.

Point to one of entries in one of the tables, for example, "63" in the multiplication table.

T: What number story could you form from this entry and "< 200"?

S: "7 × 9 < 200".

Write this information about true number stories ending with "< 200" to one side on the board.

true for = ⑥
true for <200 ㊱

T: Are there more true number stories we could form?

S: Yes, there are more using "<".

T: Could we use the tables to help us find them?

P 2

S: We could look for all the numbers smaller than 100 and use the results in true number stories ending with "< 100".

Lead the discussion until a similar observation is made. Then write this problem in the information list about true number stories.

true for $=$ ⑥
true for <200 ㊱
true for <100 (?)

T: Look at the tables and find results which could be used in true number stories ending with "< 100".

Accept a few examples from students such as 29 in the addition table corresponds to $20 + 9 < 100$.

T: How many number stories are true for < 100?

Give the students a few minutes to study the tables.

S: 33.

T: How did you find 33?

S: All except three numbers in these tables are smaller than 100 and $36 - 3 = 33$.

true for $=$ ⑥
true for <200 ㊱
true for <100 ㉝

Continue in a similar manner to find how many number stories are true for <40, <28, and <11. Add the information to your list.

true for = ⑥
true for <200 ㊱
true for <100 ㉝
true for < 40 ㉗
true for < 28 ㉓
true for < 11 ⑨

T: Are there other true number stories we could form?

S: No, we have all of them for = and all of them for <.

T: How many true number stories are there in all? Do the calculation on your paper.

Give the students a minute or two to do the calculation.

S: 134; 6 + 36 + 33 + 27 + 23 + 9 = 134.

T: Now we know how many true number stories there are. How do we find the probability of getting a true number story?

S: We need to find how many number stories there are, both the true ones and the false ones.

T: If you know how many number stories there are altogether, write it on your paper and think about how to explain your answer.

P2

Give the students a few minutes to think about this problem. Some may already know the answer from a discussion of this same question in Lesson P 1.1.

S: 360.

T: How did you find 360?

S: 3 x 3 x 4 x 2 x 5 = 360.

S: From the first three boxes there were 36 (3 x 3 x 4 = 36.) number names we could form. With each of these number names we can put "=" or "<", so there are 72 (36 x 2 = 72) expressions from the first four boxes. Then for each of these 72 expressions, there are 5 ways (numbers) to complete the number story, so there are 360 (72 x 5 = 360) number stories.

NOTE: You may need to direct the students' observations as follows.

- Use Worksheet P02 to note 36 possibilities from the first three boxes (12 from each table and 3 x 12 = 36).
- After each of the 36 possibilities from the first three boxes, use "=" or "<". That gives 36 possibilities with "=" and 36 possibilities with "<", or 72 possibilities together.
- After each of the 72 possibilities of expressions formed from the first four boxes, complete the number story with a number from the fifth box. That gives: 72 number stories ending with "11";
  72 number stories ending with "28";
  72 number stories ending with "40";
  72 number stories ending with "100";
  72 number stories ending with "200".

Altogether there are 5 x 72 or 360 number stories.

Record this information on the board.

134 true number stories
360 number stories in all

T: What is the probability of getting a true number story?

S: 134 chances out of 360.

S: $\frac{134}{360}$.

T: Can you estimate the chances with smaller numbers? Is it about 1 chance out of 2; that is, is the probability about $\frac{1}{2}$?

S: The chances are less than 1 out of 2.

S: 1 chance out of 2 would be the same as 180 chances out of 360.

S: The chances are closer to 1 out of 3.

S: 1 chance out of 3 would be the same as 120 chances out of 360, so it is a little better than 1 chance out of 3.

Allow the class to discuss this situation, but do not insist on such precise observations.

Save the boxes for use in Lesson P 1.3.

P 6

1.3 Random Number Stories #3

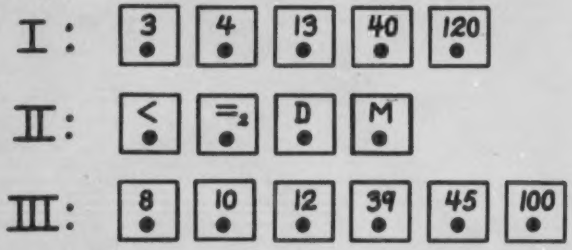
CAPSULE LESSON SUMMARY

Examine number stories that can be formed by selecting one symbol from each of three boxes where the first box contains five numbers, the second box contains four relations, and the third box contains six numbers. Discuss the probability of getting a true number story when random selection is made. (Optional) Pose a problem in a similar situation of determining which numbers to put into the boxes so as to have the largest (smallest) possible probability of getting a true number story.

MATERIALS

Teacher: Three boxes; pieces of paper (about 10 cm by 10 cm)
Student: Paper; colored pencils; Worksheets P03, P04, P05, P06

NOTE: Prior to presentation of this lesson, prepare three boxes with labels "I", "II", and "III". Into the boxes, put pieces of paper marked as indicated below. Fold the pieces of paper so as to hide what is written on them.



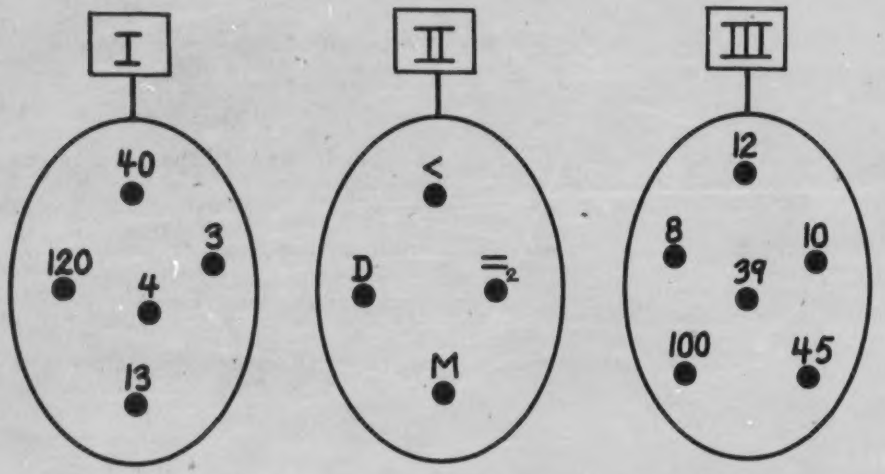
Worksheets P05 and P06 are answer keys for Worksheets P03 and P04. The correctly filled in tables on Worksheets P05 and P06 are used to answer some questions towards the end of Exercise 1. You might wish to remove these worksheets from the students' booklets and distribute them at the proper time.

DESCRIPTION OF LESSON

Exercise 1

T: In these three boxes there are pieces of paper, each with a symbol written on it and folded so that the symbol is hidden. The picture I draw will show what is in the boxes.

As you draw this string picture on the board, read the symbols and describe what is in each box (string).



Review or explain the meaning of the relation symbols, giving examples and letting students find examples. Be sure the class understands that "D" is an abbreviation for "is a divisor of" and "M" is an abbreviation for "is a multiple of". To explain $=_2$ (read "equals modulo 2"), remind the class about the relation $=_3$, which is similar. Then present some examples.

$$\begin{aligned} 8 &=_2 14 \\ 15 &=_2 37 \\ 3 &=_2 101 \\ 54 &=_2 26 \end{aligned}$$

T: Who can explain the meaning of "$=_2$"?

S: Two numbers are equal modulo 2 if and only if they are both even or both odd.

P 6

S: There are two strings, one for the multiples of 2 (even numbers) and one for the multiples of 2 plus 1 (odd numbers). "$=_2$" means "are in the same string".

Ask students to give some of their own examples of numbers which are equal modulo 2 and examples of numbers which are not equal modulo 2.

T: Now we will perform trials of an experiment with these three boxes to make number stories. I'll shake each of the boxes and one of you will select, without looking, one piece of paper from each box.

Choose a student to make the selection starting with the first box. Ask the student to read aloud what is on the piece of paper immediately after it is selected and you record the result on the board.

S: 3 . . . D.

T: Wait a moment before you select from the third box. Can we predict whether the number story will be true or false?

S: Both are possible.

T: Which is more likely?

S: They both have the same chance. There are three chances out of six that it will be true, because 3 is a divisor of 12, 39, and 45. Also there are three chances out of six that it will be false, because 3 is not a divisor of 8, 10, and 100.

Permit the student to select from the third box and write the result on the board. For example,

3 D 10 F

Be sure the pieces of paper are refolded and returned to their correct boxes.

Choose another student to make a selection this time starting with the third box and then going on to the second box.

S: < . . . 45.

T: Wait a moment before you select from the first box. Can we predict whether the number story will be true or false? Which is more likely?

S: Both are possible, but a true number story is more likely.

S: There are four chances out of five that it will be true (3, 4, 13, and 40 are smaller than 45). There is only one chance out of five that it will be false (120 is not smaller than 45).

Permit the student to select from the first box and write the result on the board. For example,

4 < 45 T

Let students perform two or three more trials of the experiment with the boxes, alternating between starting the selection with the first box or the third box. List the number stories on the board; for example,

3 D 10 F
4 < 45 T
40 $=_2$ 100 T
3 D 39 T
13 $=_2$ 10 F

Without performing more trials, ask the class to consider other examples of number stories that you could get.

T: Who can give us an example of a true number story we could get with M?

P 6

S: "40 M 8".

Write the example in your list on the board.

40 M 8	T

A common mistake in this situation is to confuse M with D. When this happens, write both the false and true number stories on the board. For example,

4 M 8	F
4 D 8	T

Ask the students to find more examples of true and false number stories from this situation and to write them on their paper. Encourage them to try to find at least one true and one false number story for each relation.

After several minutes of independent work, record some of the student's examples in your list on the board.

3 D 10	F
4 < 45	T
40 $=_2$ 100	T
3 D 39	T
13 $=_2$ 10	F
40 M 8	T
13 < 12	F
120 $=_2$ 45	F
3 < 100	T
13 M 10	F
4 M 8	F
4 D 8	T

T: We can get both true and false number stories for each relation. Considering all the possible number stories we could get, do you think it is more likely that we get a true one or a false one?

S: False. I think there are more false number stories than true ones.

T: How could we find out?

S: Look at all of the possible number stories and check which ones are true and which are false.

T: How many possible number stories are there?

Some students may recognize that the multiplication rule applies to this counting problem and, therefore, answer that there are 5 x 4 x 6 or 120 possible number stories. If not, continue by assuring the class that further investigation will help to answer this question. In any case, go on to consider specific kinds of number stories.

T: Let's look first at the number stories using "$=_2$". Do you know how many there are?

S: 30.

T: A table will make it easier to look at all the number stories using "$=_2$".

P 6

Begin to draw this table on the board. Explain as you go that the numbers in the first box (string) are arranged vertically under "$=_2$" and that the numbers in the third box (string) are arranged horizontally to the right of "$=_2$".

$=_2$	8	10	12	39	45	100
3						
4						
13						
40						
120						

Direct the class to locate in the table the spaces for any examples in your list using "$=_2$". Then write "T" or "F" as is appropriate for the particular number story.

$=_2$	8	10	12	39	45	100
3						
4						
13		F				
40						T
120					F	

T: How does this table help us to count the number stories using "$=_2$"?

S: There are five rows and six spaces in a row, so there are 5 x 6 or 30 spaces, 30 number stories using "$=_2$".

T: To make it easier to see the true number stories, we will use a red dot instead of a "T". Then for a false number story, we will leave a blank space.

Erase the "F"'s and replace the "T"'s by red dots in your table.

$=_2$	8	10	12	39	45	100
3						
4						
13						
40						●
120						

Distribute Worksheets P03 and P04 and ask the students to go on by themselves finding the true number stories.

When many students have finished filling in the tables, continue the full group discussion.

T: Using these tables, can you tell how many possible number stories there are?

P 6

S: There are 30 represented in each table and there are 4 tables, so there are 4 × 30 or 120 possible number stories.

If some students have not completely filled-in the tables or have made many mistakes, distribute the correctly filled in tables on Worksheets P05 and P06.

T: How many true number stories using "$=_2$"?

S: 16.

T: And using "<"?

S: 17.

T: And using "D"?

S: 7.

T: And using "M"?

S: 5.

Write this information on the board.

120 Number Stories

true for $=_2$	⑯
true for $<$	⑰
true for D	⑦
true for M	⑤

T: How many true number stories are there?

S: 45 (16 + 17 + 7 + 5 = 45).

T: How many false number stories are there?

S: 75 (120 - 45 = 75).

S: So it is more likely to get a false number story.

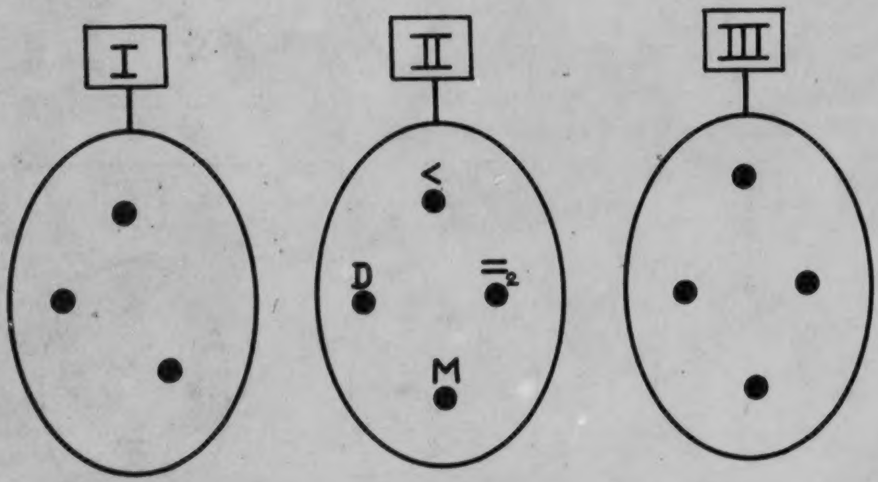
T: What is the probability of getting a true number story?

S: 45 out of 120 (or $\frac{45}{120}$).

Direct the class to observe that $\frac{45}{120} = \frac{3 \times 15}{8 \times 15} = \frac{3}{8}$.

Exercise 2

NOTE: This is an optional exercise which you may do if time permits. Draw this string picture on the board.

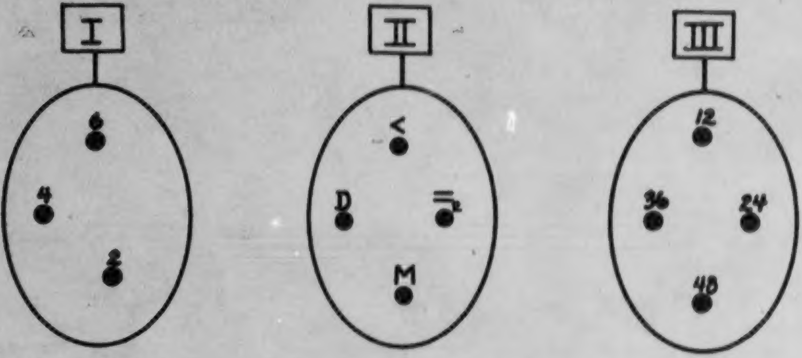


P 6

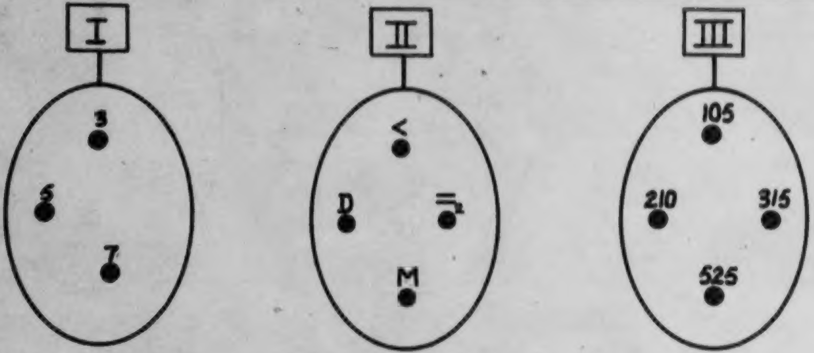
T: Imagine the three boxes. In the second box, we put the same relation symbols as before. In the first box we put three numbers, and in the third box we put four numbers. Your problem is to decide which numbers to put in the first and third boxes so as to have the largest probability of getting a true number story.

Instruct students to copy the picture on their papers, then to label the dots with positive whole numbers (all different) so as to have as many true number stories as possible.

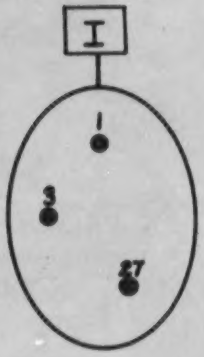
Let students work independently. As labelings are found, direct students to calculate the probability. Encourage using tables to help count the true number stories. Record some of the probabilities students find on the board. Here are some solutions proposed by students in our developement classes.

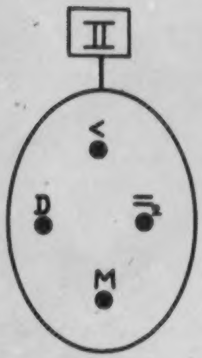


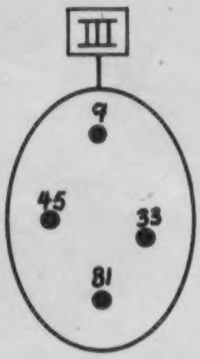
Probability of getting a true number story = $\frac{36}{48} = \frac{3}{4}$.



Probability of getting a true number story = $\frac{36}{48} = \frac{3}{4}$.



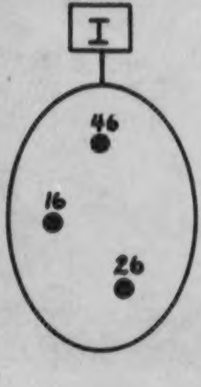


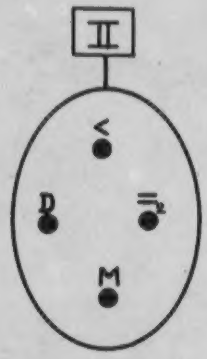


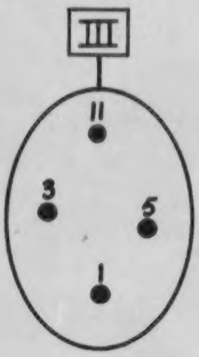
Probability of getting a true number story = $\frac{33}{48} = \frac{11}{16}$.

NOTE: Since D and M (or < and M) are mutually exclusive (i.e., a red dot in the table for D means a blank space in the table for M and visa - versa), the D and M (or the < and M) tables together can have at most one out of two true number stories. This means that for the four tables together representing all the number stories, the largest possible probability of getting a true number story is $\frac{3}{4}$ (3 out of 4).

Students who finish this problem quickly could be asked to label the dots so as to have the smallest possible probability of getting a true number story. For this problem, a labeling such as the following makes it impossible to get a true number story.







Worksheet P05 is an answer key for Worksheet P03.

Name ____________________ P 05

<	8	10	12	39	45	100
3	●	●	●	●	●	●
4	●	●	●	●	●	●
13				●	●	●
40					●	●
120						

$=_2$	8	10	12	39	45	100
3				●	●	
4	●	●	●			●
13				●	●	
40	●	●	●			●
120	●	●	●			●

Worksheet P06 is an answer key for Worksheet P04.

Name ____________________ P 06

D	8	10	12	39	45	100
3			●	●	●	
4	●		●			●
13				●		
40						
120						

M	8	10	12	39	45	100
3						
4						
13						
40	●	●				
120	●	●	●			

2

PROBABILITY GAMES

The lessons of this chapter deal with a game for two people played with marbles. Two marbles, one red and one blue, are used and from them one marble is chosen at random. If the red marble is chosen, player A scores a point and if the blue marble is chosen, player B scores a point. The player to accumulate 10 points first is the winner.

A geometric, measure-minded method is used to analyze situations where an intermediate score is known and both players' chances for winning is to be computed. The problems are essentially the same as those in which one must decide how to share the stake of a probabilistic situation according to the probabilities involved. The new method is a generalization of the combinatorial or "tree" method. Furthermore, it provides another opportunity for students to explore patterns in sequences of numbers.

CAPSULE LESSON SUMMARIES

Section 2.1 :	Introduce a probability game played by two people. Follow the game on a specially drawn grid and observe how to use the grid to read scores and to determine the winner. Calculate the chances of the individual players winning when the score is not tied. (Worksheet P07)
Section 2.2 :	Review the probability game introduced in Section 2.1. Calculate the winning chances for the two players given different intermediate scores. Record these chances on a chart and observe some patterns which make the calculations simpler. (Worksheets P07, P08)

Section 2.3: Review the probability game from Sections 2.1 and 2.2 and recall how to read the posters. Investigate some patterns in the chart on Poster #2 and use these patterns to complete the chart. Use the completed chart to discuss and solve some problems about related probability games. (Worksheets P08, P09)

2.1 Probability Games #1

CAPSULE LESSON SUMMARY

Introduce a probability game played by two people. Follow the game on a specially drawn grid and observe how to use the grid to read scores and to determine the winner. Calculate the chances of the individual players winning when the score is not tied.

MATERIALS

Teacher:	Probability Posters #1 and #2 for IG-IV; Scorecards for Probability Games; colored chalk; one red and one blue marble
Student:	Paper; colored pencils; Worksheet P07

NOTE: If you have a magnetic chalkboard available, prepare the scorecards by sticking a small piece of magnetic material (provided in the classroom set of materials) on the back of each card. Otherwise, prepare scorecards to stick to the chalkboard by attaching a loop of masking tape or a small wad of plastic caulking compound (e.g. Mortite or Rope Caulk) to the back of each card. These score cards are needed in Exercise 2.

DESCRIPTION OF LESSON

Display Probability Poster #1 for IG-IV at one end of the board and Probability Poster #2 for IG-IV at the other end of the board. Distribute Worksheet P07 on which there is a copy of Poster #1.

Exercise 1

Write this information on the (center) board.

● Rita — ● Bruce

Use one red marble and one blue marble to demonstrate the game being played by Rita and Bruce as you describe it.

P 3

T: Rita and Bruce are playing a game. Rita has one red marble and one blue marble. Keeping her hands behind her back, she mixes them and then puts one marble in each hand. Bruce chooses a hand. If he chooses the hand with the blue marble, he scores a point. If he selects the hand with the red marble, Rita scores a point. Remember, blue for Bruce and red for Rita.

Demonstrate the game with the two marbles by keeping your hands behind your back while you mix up the marbles and place one marble in each hand. Then invite a student to select a hand. Record the result by drawing one red dot or one blue dot on the board according to the color of the marble chosen. Repeat this process about five times. For example, if the marbles appeared in this order: red, blue, blue, red, blue, you would draw the following dots.

● ● ● ● ●

The ensuing discussion will depend on the results obtained in your class. It might be as follows.

T: What is the score now?

S: Two points for Rita and three points for Bruce.

Record this score on the board.

● Rita — ● Bruce
2 — 3

T: The score is 2 - 3 (read this score as "2 to 3"). We will always say Rita's score first. The winner of this game is the first person to score ten points.

Continue playing the game with the class. Perhaps the blue marble will be chosen next. Record this by adding a blue dot to the line of dots on the board.

● ● ● ● ● ●

T: What is the score now?

S: 2 to 4.

Insist that the student announces Rita's score first, i.e., the score is now 2 - 4, not 4 - 2. Replace "2 - 3" with "2 - 4" on the board.

● Rita — ● Bruce
2 — 4

T: The score is now 2 - 4. What could the next score be?

S: 3 - 4, if the red marble is chosen.

S: 2 - 5, if the blue marble is chosen.

Again, mix the marbles behind your back and invite a student to choose a hand. The red marble might be chosen. Draw another red dot on the board.

● ● ● ● ● ● ●

T: What is the score now?

S: 3 - 4.

Replace "2 - 4" with "3 - 4" on the board.

● Rita — ● Bruce
3 — 4

P 3

T: What could the next score be?

S: 4 - 4 or 3 - 5.

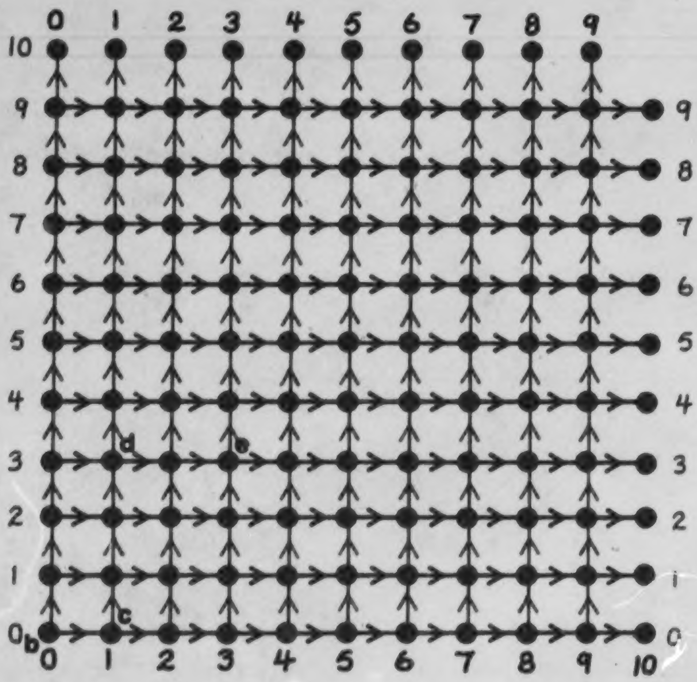
T: Yes, the score would be 4 - 4 if the red marble were chosen and it would be 3 - 5 if the blue marble were chosen.

Continue this activity until you feel your class understands the rules and the scoring system of the game. If the class insists, you could play until there is a winner.

T: Do you think that this is a fair game?

S: Yes, Rita and Bruce both start at zero and try to get ten. On each turn, there is an equal chance for the red marble or the blue marble since there is one of each.

Call the class' attention to Poster #1.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: You have a copy of this grid on Worksheet P07. Every game Rita and Bruce play can be shown on this grid. Follow along on your worksheet as I trace a path on the poster. (Point to b, the dot for 0 - 0.) Each game starts at 0 - 0. (Trace the red arrow from b to c.) If the red marble is chosen first, what is the score?

S: 1 - 0.

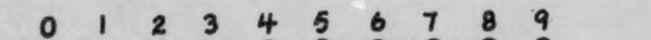
T (tracing three blue arrows starting at c and going upward to d): If the blue marble is chosen three times in a row, what is the score?

S: 1 - 3.

T (tracing two red arrows starting at d and going to the right to e): If the red marble is then chosen two times, what is the score?

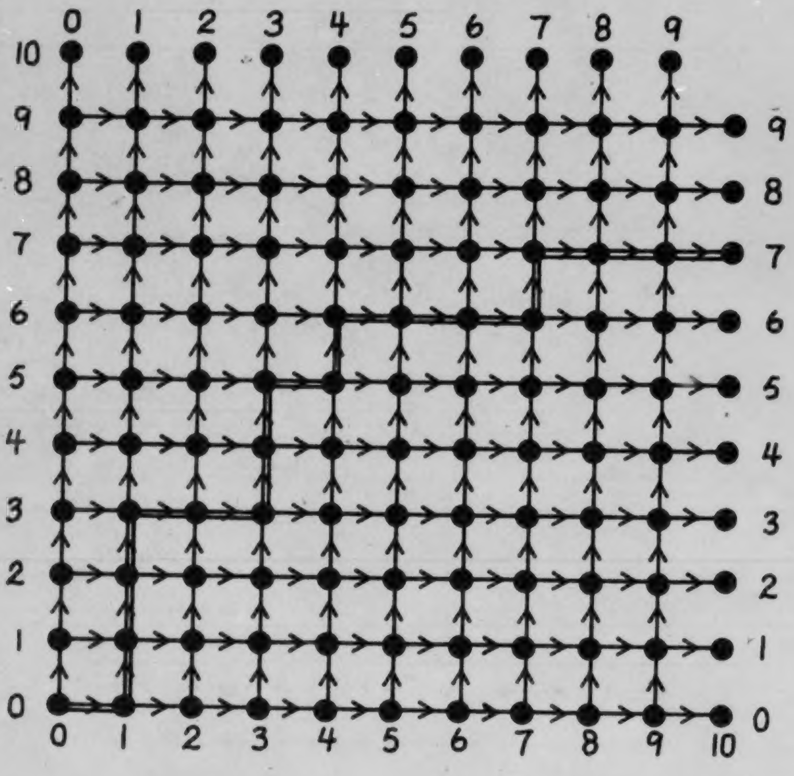
S: 3 - 3.

Continue to describe one of the possible games, for example, the game illustrated below. A game is finished when you reach a blue dot in the top row (a win for Bruce) or a red dot in the rightmost column (a win for Rita).



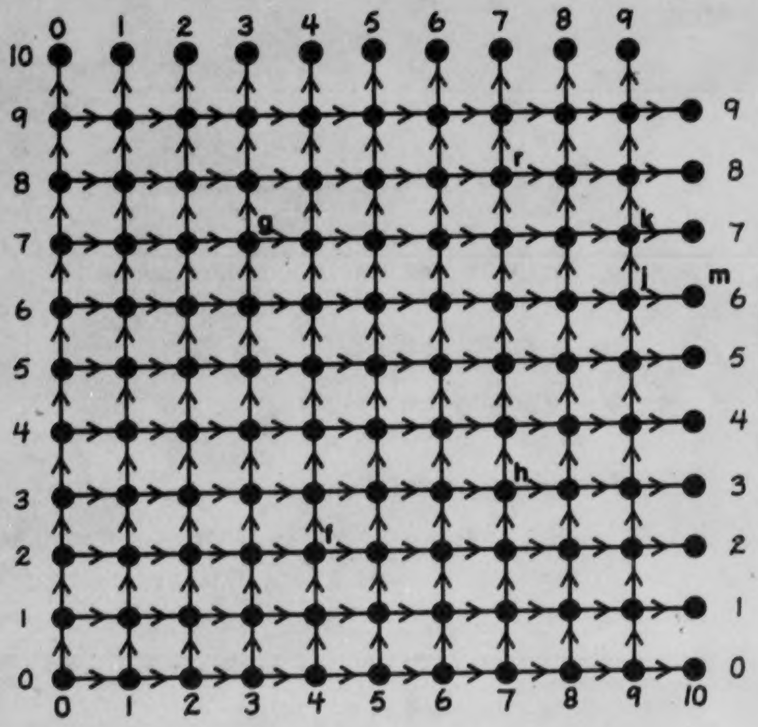
P 3

Invite a student to trace arrows on the grid to represent any complete game which is won by Rita. The class should check that the student starts at the dot for 0 - 0, follows red arrows to the right or blue arrows upward, and ends at a red dot in the rightmost column. For example, a student might trace the path drawn in black in the following illustration.



Similarly, invite a student to trace a path from 0 - 0 to simulate a game that Bruce wins. In this case, the path must end at a blue dot in the top row. (See, for example, the illustration on page 41.)

Next, point to specific dots on the grid and ask what scores are represented by the dots. For example,



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to f): What score does this dot represent?

S: 4 - 2, four points for Rita and two points for Bruce.

Illustrate how to read the score on the grid. From f, trace downward to "4" for Rita's score and then trace to the left to "2" for Bruce's score.

After you have asked for the scores represented by several dots, reverse the question by asking the students to locate dots which represent specific scores. For example,

T: Which dot represents a score of 3 - 7?

A student should point to g. If a student suggests h, the dot for 7 - 3, remind the class that Rita's score is always given first and that her score of 3 is represented by "3" along the horizontal axis.

T: Which dot is for a score of 9 - 6?

A student should point to j.

T: If the score of a game is 9 - 6, what could the next score be?

S: 10 - 6 and a win for Rita if the red marble is chosen, or 9 - 7 if the blue marble is chosen.

Use the grid to demonstrate how to observe this answer.

T: If the blue marble is chosen (trace a blue arrow from j to k), the score changes from 9 - 6 to 9 - 7. If the red marble is chosen (trace a red arrow from j to m), the score changes from 9 - 6 to 10 - 6 and Rita wins.

Repeat this exercise by asking a student to find the dot for 7 - 8 (r) and to show on the grid what the next score could be (8 - 8 or 7 - 9).

Exercise 2

T: One afternoon Rita and Bruce played this game until they had to stop. But the game was not over; they stopped when the score was 7 - 8 and agreed to continue the next day. When they continue the game the next day, who is favored to win?

S: Bruce, because he is ahead.

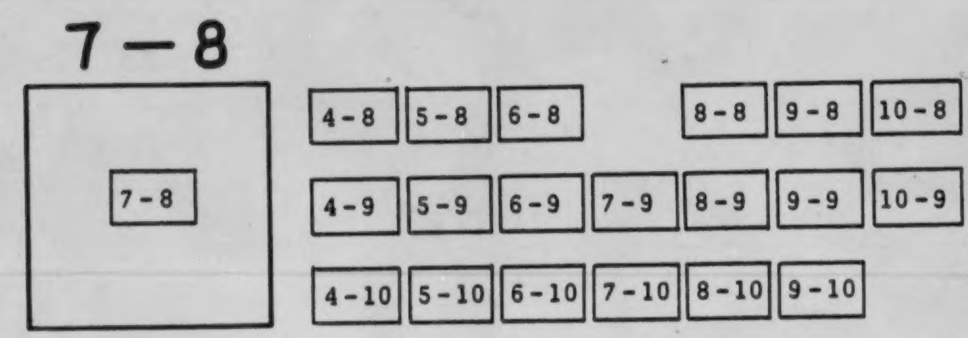
T: Does Bruce have a large advantage?

S: No.

T: Bruce's probability of winning is better than $\frac{1}{2}$. What do you estimate his chances to win are?

Accept any student suggestions which show an advantage for Bruce, such as a probability of $\frac{2}{3}$ (2 chances out of 3) or $\frac{3}{5}$ (3 chances out of 5).

Draw a large square on the board. Place the scorecard $\boxed{7-8}$ in the square and display the other scorecards to one side of the board. Write "7 - 8" near the square.

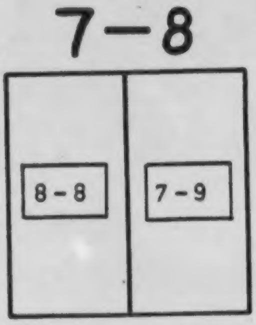


T: We will use this square to calculate Bruce's chances of winning when the score is 7 - 8. If the score is 7 - 8, what could the next score be?

S: 8 - 8 if the red marble is chosen or 7 - 9 if the blue marble is chosen.

T: There is one red marble and one blue marble. Therefore each of these two scores is equally likely. I will show this on the square by dividing it in half.

Dividethe square in half and place the two scorecards as shown.

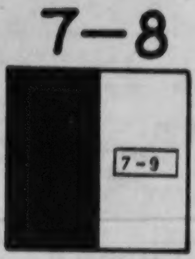


P 3

T: Now we must consider these two games, one with a score of 8 - 8, the other with a score of 7 - 9. If the score in a game is 8 - 8, who is more likely to win, Rita or Bruce?

S: It is a tie game at 8 - 8, therefore their chances are equal.

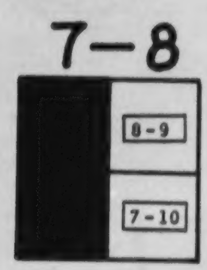
T: I will indicate this by coloring one-half of the region for 8 - 8 red and the other half of it blue.



T: If the score is 7 - 9, what could the next score be?

S: 8 - 9 if the red marble is chosen or 7 - 10 if the blue marble is chosen.

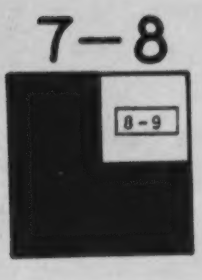
T: Each of these two scores, 8 - 9 and 7 - 10, is equally likely. Therefore, I will divide the region for 7 - 9 in half. I will put $\boxed{8 - 9}$ in one half and $\boxed{7 - 10}$ in the other half.



T: What do we know if the score in a game is 7 - 10?

S: Bruce has won.

T: Therefore, I'll color the region for 7 - 10 blue.

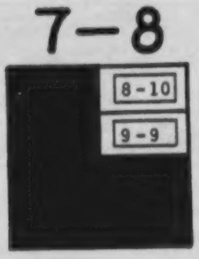


T: If the score is 8 - 9, what could the next score be?

S: 9 - 9 or 8 - 10.

T: How can I show this?

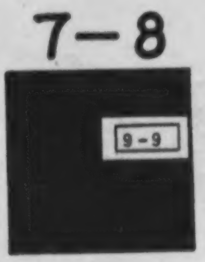
S: Divide the region for 8 - 9 in half. Put 9 - 9 in one half and 8 - 10 in the other half.



T: What happens if the score is 8 - 10?

S: Bruce wins.

S: Color the region for 8 - 10 blue.



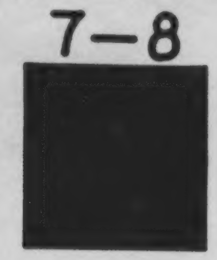
P 3

T: If the score is tied at 9 - 9, what do we know about Rita's and Bruce's chances of winning the game.

S: Rita's and Bruce's chances are equal.

T: How can I show this?

S: Divide the 9 - 9 part of the square in half and color one half of the region red and color the other half blue.

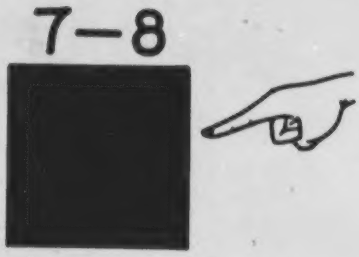


T: Starting when the score in a game is 7 - 8, the blue region represents Bruce's chances for winning and the red region represents Rita's chances for winning. From the picture, can we tell who is more likely to win?

S: Bruce, because more of the square is blue than red.

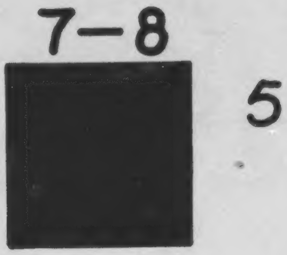
T: How can we calculate the chances of winning for Bruce and Rita?

Invite the class to discuss this question. Encourage any suggestions about comparing the areas shaded red and blue. Lead the class to a method as follows.



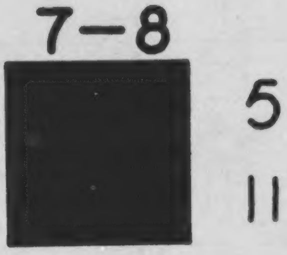
T (pointing to the smallest red region): One method to calculate the chances for Bruce and Rita is to count how many pieces of this size fit into the entire blue region and how many fit into the entire red region. How many pieces of this size could fit into the red region?

Accept the answers of several students. Then invite a student with the correct answer of five to explain this answer. It may be useful to subdivide the larger red region to show how four small pieces could fit into it. Then write "5" in red near the square.



T: How many pieces of the same size fit into the blue region?

First let the students study the picture and suggest answers. Then invite students to explain their methods of counting until the correct answer of eleven is agreed upon. You may wish to subdivide the blue region to show the answer more explicitly. Then write "11" in blue near the square.

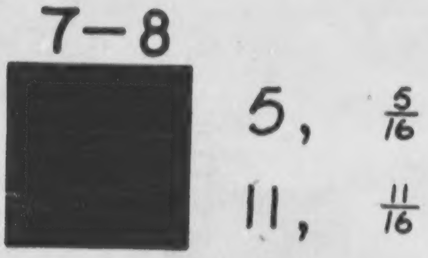


T: Therefore, if the score in a game is 7-8, Bruce has 11 chances of winning and Rita has 5 chances. As expected, Bruce is favored. In fact, Bruce is more than twice as likely to win since 11 is more than 2 × 5.

P 3

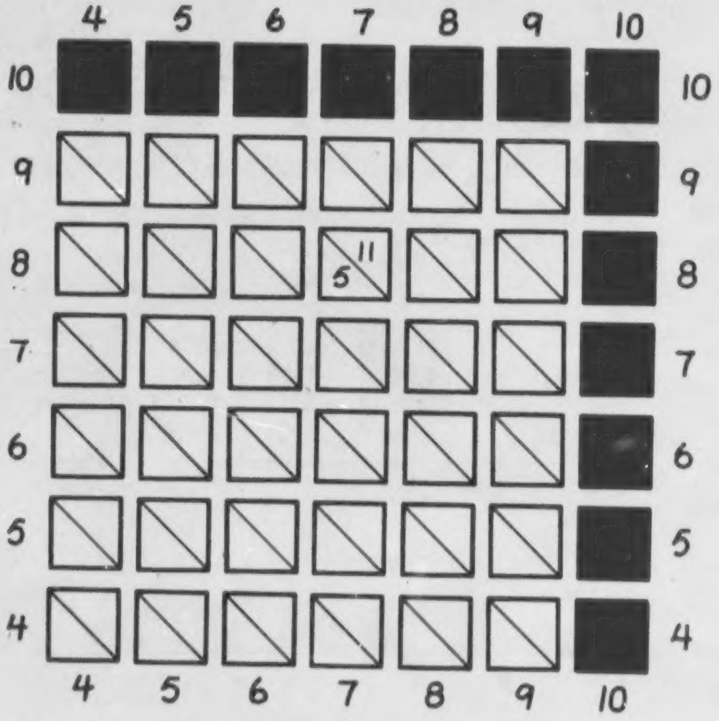
T: We can also calculate Rita's and Bruce's probabilities to win. Since 5 + 11 = 16, Rita has 5 chances out of 16 to win and Bruce has 11 chances out of 16 to win. These probabilities can be written as fractions.

Near the square, write these probabilities as fractions.


7–8
5, $\frac{5}{16}$
11, $\frac{11}{16}$

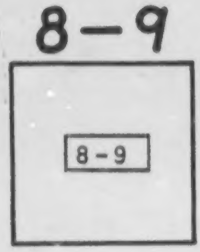
Encourage the class to discuss the method of solution and the result.

Record the result on Poster #2 as shown below.


4 5 6 7 8 9 10
10 9 8 7 6 5 4
11
5
10 9 8 7 6 5 4
4 5 6 7 8 9 10

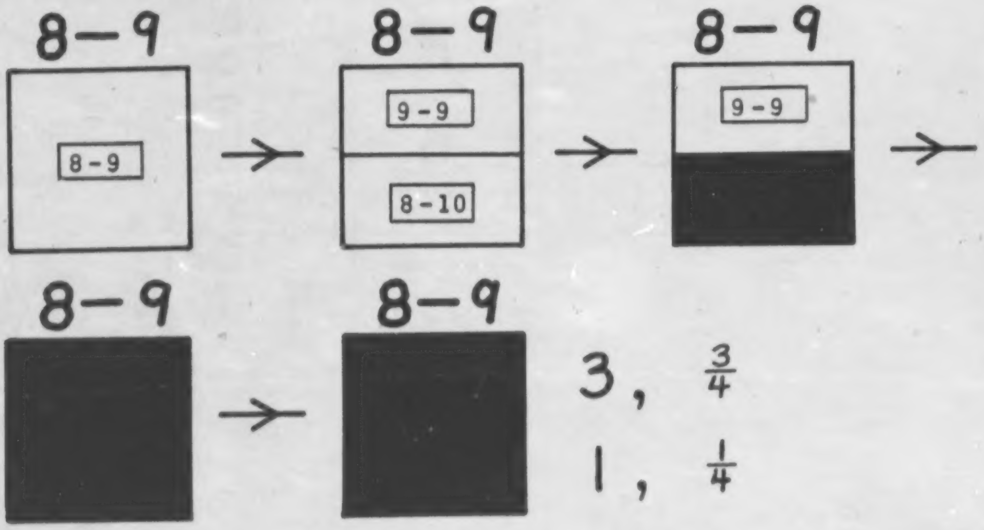
Indicate that you have put the results for the score 7 - 8 in the square which is in the column labeled "7" (for Rita's score) and the row labeled "8" (for Bruce's score).

Erase the board and draw this picture.



T: On another day, the game was stopped when the score was 8 - 9. Draw a square on your paper and try to calculate the winning chances for Bruce and Rita when the score is 8 - 9. Your method of solution can be similar to the method we just used.

As you let the students work independently for a few minutes, assist any students who need help. Then use student suggestions to solve the problem collectively at the board. The squares drawn below show one procedure to calculate the chances. Your class might divide and color the square in a different way, but they should arrive at the same result.



P 3

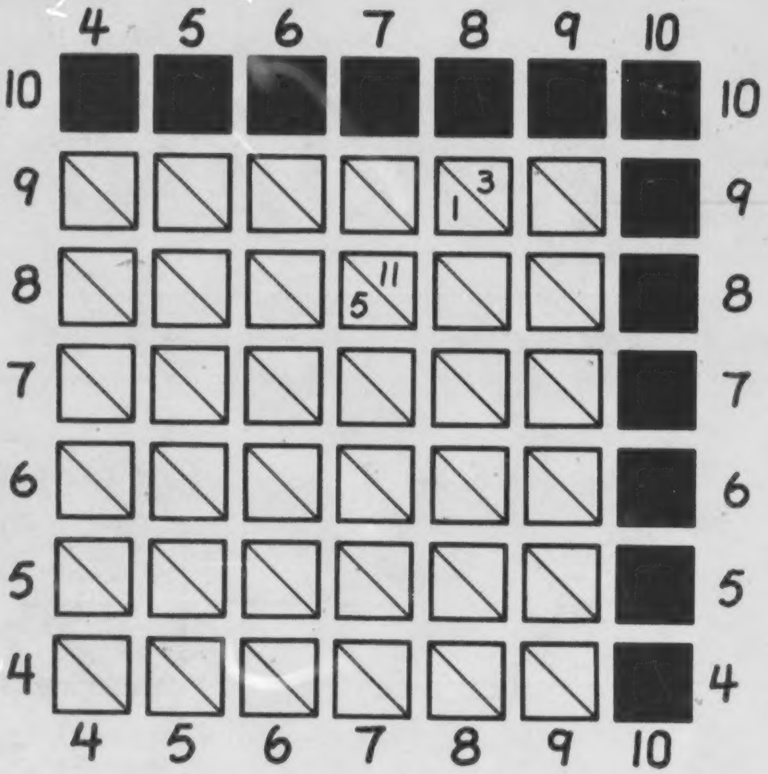
T: Who is favored to win if the score is 8 - 9?

S: Bruce.

T: How much more likely is Bruce to win than Rita?

S: Three times.

Invite a student to record this result for 8 - 9 on the chart on Poster #2. Explain to the class that the result for 8 - 9 is placed in the column labeled "8" and the row labeled "9".



T: If we know the winning chances for Rita and Bruce when the score is 8 - 9, do we also know the results for another score?

S: Yes, for 9 - 8.

T: What are the chances when the score is 9 - 8?

S: Rita has three chances and Bruce has one chance.

T: Why?

S: Since there is one red marble and one blue marble, the advantage for Bruce when the score is 8 - 9 is the same as Rita's advantage when the score is 9 - 8.

Invite a student to record the result for 9 - 8 in Poster #2 as shown in the next illustration.

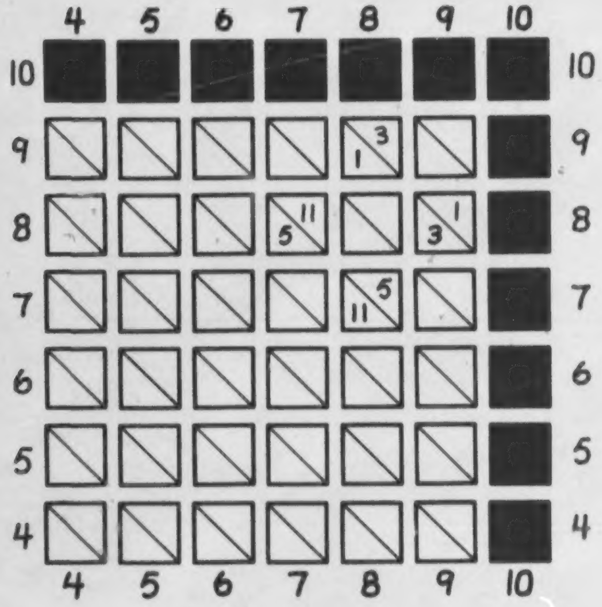
T: Do we know the result for another score?

S: Yes, 8 - 7, because we already know the result for 7 - 8.

T: What are the chances for Rita and Bruce when the score is 8 - 7?

S: Eleven chances for Rita and five chances for Bruce.

Invite a student to record this result for 8 - 7 on Poster #2.



P 3

Collect Worksheet P07 and save the worksheets and both posters for use in Lesson P2.2.

2.2 Probability Games #2

CAPSULE LESSON SUMMARY

Review the probability game introduced in Lesson P2.1. Calculate the winning chances for the two players given different intermediate scores. Record these chances on a chart and observe some patterns which make the calculations simpler.

MATERIALS

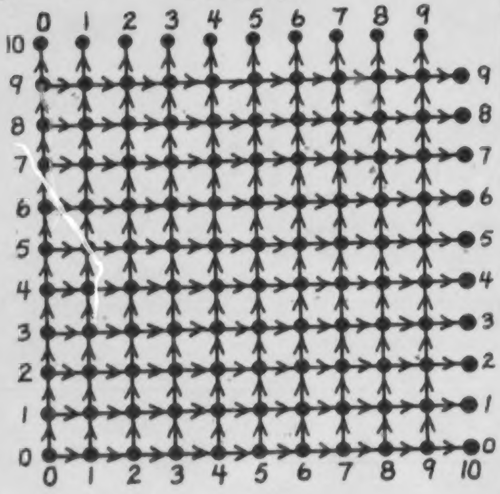
Teacher: Probability Posters #1 and #2 for IG-IV, Scorecards for Probability Games; one red and one blue marble

Student: Paper; colored pencils; Worksheets P07, P08

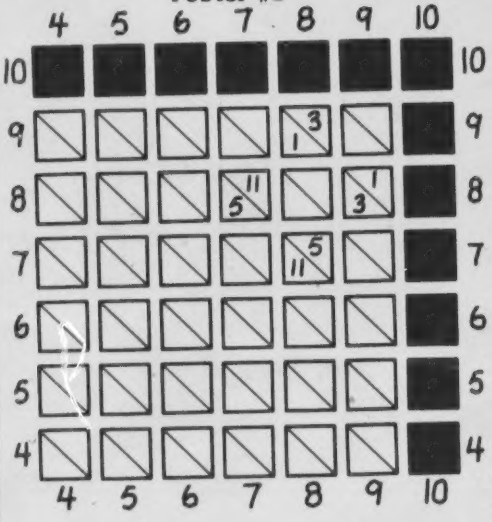
NOTE: Scorecards should be prepared as described for Lesson P2.1 so that they stick to the board.

DESCRIPTION OF LESSON

Display Probability Posters #1 and #2 for IG-IV at opposite sides of the board. The results from Lesson P2.1 should still be recorded on Poster #2. Place the scorecards near the center of the board.



Poster #1



Poster #2

4 - 8	5 - 8	6 - 8	7 - 8	8 - 8	9 - 8	10 - 8
4 - 9	5 - 9	6 - 9	7 - 9	8 - 9	9 - 9	10 - 9
4 - 10	5 - 10	6 - 10	7 - 10	8 - 10	9 - 10	

P4

Distribute Worksheets P07 and P08. Worksheet P07 was collected at the end of Lesson P2.1.

T: Who remembers the game Rita and Bruce played with marbles?

The students should mention at least the following aspects of the game:

- one red marble and one blue marble are used;
- Rita scores one point if the red marble is chosen and Bruce scores one point if the blue marble is chosen;
- the first player to score ten points wins.

You may wish to use a red marble and a blue marble to demonstrate the game a few times.

Write this information on the board.

● Rita — ● Bruce
7 — 3

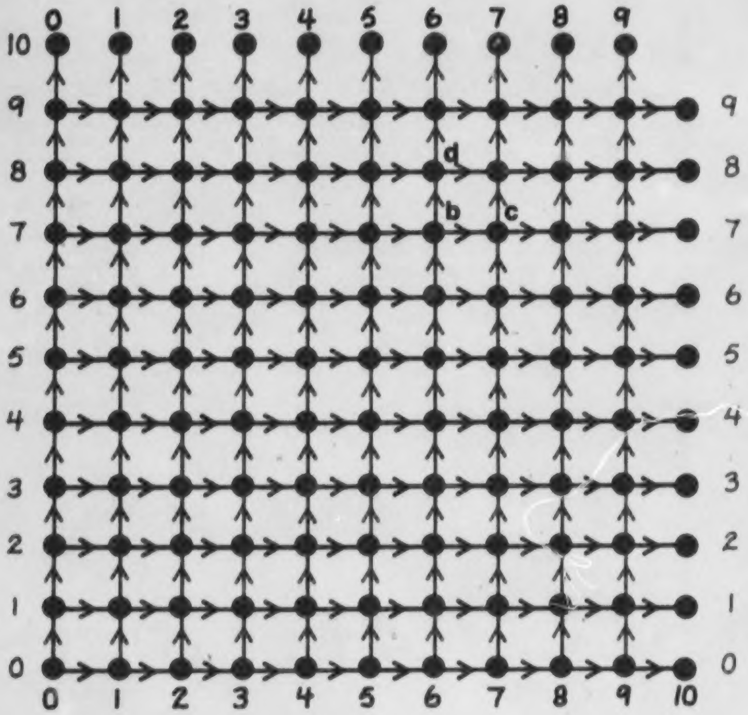
T: If the score in a game is 7-3 (read this as "7 to 3"), what could the next score be?

S: 8-3 if the red marble is chosen or 7-4 if the blue marble is chosen.

Erase "7-3" and write "9-7". Ask what the next score could be. (10-7 and a win for Rita or 9-8)

Erase "9-7", write "6-7", and ask what the next score could be. (7-7 or 6-8)

Call the class' attention to the red-blue grid on Poster #1.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Which dot represents a score of 6-7?

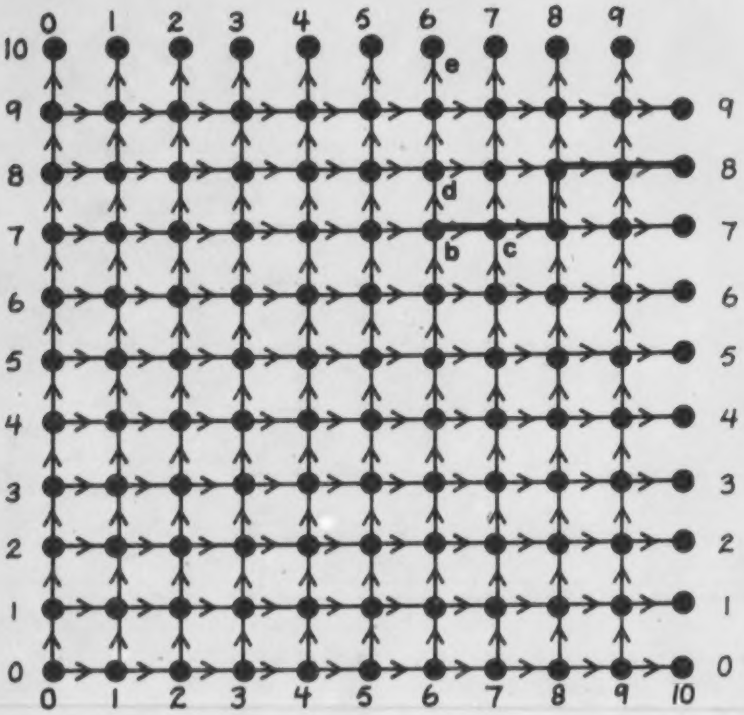
S: This dot (b) is for 6-7 because it is in the column marked "**6**" for Rita's score and the row marked " 7 " is for Bruce's score.

T: The grid shows that the score changes from 6-7 to 7-7 if the red marble is chosen (trace a red arrow from b to c). The score changes from 6-7 to 6-8 if the blue marble is chosen (trace a blue arrow from b to d).

If the score in a game is 6-7, who can trace a path to show how Rita could win the game?

The student's path should start at b, follow blue arrows upward or red arrows to the right, and end at a red dot in the rightmost column. For example, a student might trace this path shown in black.

P 4



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: If the score is 6-7, who can trace a path to show how Bruce could win the game?

The student's path should start at b, follow red arrows to the right or blue arrows upward, and end at a blue dot in the top row.

T: Starting at 6-7, trace a path showing the shortest possible conclusion to the game.

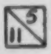
S: The shortest conclusion is a blue marble chosen the next three times. Bruce wins in three moves with the score 6-10.

Invite the student to trace the corresponding path: start at b and follow three blue arrows upward to e, the blue dot for 6-10.

Refer to the chart on Poster #2.

T: In a previous lesson, we analyzed some games and recorded the results on this chart. Who can explain what some of these numbers mean?

Accept the explanations of several students about Rita's and Bruce's chances for winning given certain scores. If necessary, ask the following questions.

T (pointing to the square "" for the score 8-7): What do the "11" and "5" in this square mean?

S: If the score in a game is 8-7, Rita has 11 winning chances and Bruce has only 5.

T: If the score is 8-9, what are Bruce and Rita's chances?

S: Bruce has 3 winning chances and Rita has 1.

T: When does Bruce have 11 chances to win and Rita 5?

S: When the score is 7-8.

Write this information on the board.

7–8: 5 11
8–9: 1 3

T: Bruce is favored for each of these scores. Which of these two scores is better for Bruce?

Encourage both intuitive answers based on the game situation and quantitative answers based on the chances written on the board.

P 4

S: 8-9 is better for Bruce, because in both situations Rita is one point behind Bruce. But when the score is 8-9, Rita has less time to catch up and win since Bruce is only one point from victory.

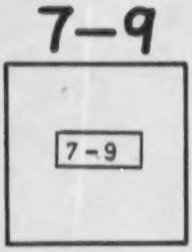
S: For 8-9, Bruce has 3 chances out of 4 to win so his probability of winning is $\frac{3}{4}$. For 7-8, Bruce has 11 chances out of 16 to win, so his probability of winning is $\frac{11}{16}$. The score is better for Bruce since $\frac{3}{4} = \frac{12}{16}$ and $\frac{12}{16}$ is larger than $\frac{11}{16}$.

S: 8-9 is the better score for Bruce, because he is three times more likely to win than Rita since $3 \times 1 = 3$. When the score is 7-8, Bruce is only about two times as likely to win as Rita since 11 is just a little larger than 2×5.

If no student gives this last explanation, present it to the class yourself.

Ask your students to copy the results already recorded on Poster #2 onto Worksheet P08.

Draw this square on the center board and place the scorecard for $\boxed{\text{7-9}}$ in it. Write "7-9" above the square.

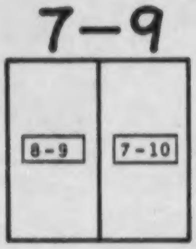


T: One day, Rita and Bruce stopped playing when the score was 7-9. Let's calculate the winning chances for each player if they continue the game the next day. If the score is 7-9, what could the next score be?

S: 8-9 if the red marble is chosen or 7-10 if the blue marble is chosen.

T: How can I show this on the square?

S: Divide the square in half. Put $\boxed{8\text{-}9}$ in one part and $\boxed{7\text{-}10}$ in the other part.

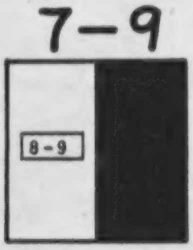


T (pointing to "$\boxed{7\text{-}10}$"): What happens if the score is 7-10?

S: Bruce is the winner.

T: How can I show this?

S: Color that half of the square blue.



T (pointing to "$\boxed{8\text{-}9}$"): What do we know if the score is 8-9?

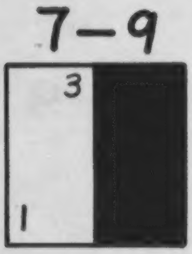
S: The next score could be 9-9 or 8-10. You should divide the region for 8-9 in half.

T: Yes, I could calculate Rita's and Bruce's winning chances when the score is 8-9 that way. But do I have to do all that work?

S: No, the results for 8-9 are already recorded on the poster. If the score is 8-9, Bruce has 3 chances to win and Rita has only 1 chance.

P 4

Invite a student to point to where this result is recorded on Poster #2. Then record it on the board.

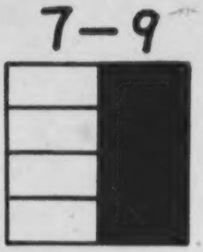


T (pointing to the uncolored region): How should I divide this region?

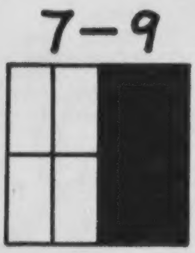
S: Into four pieces and color three parts blue and one part red.

T: Who can divide this region into four pieces of the same size?

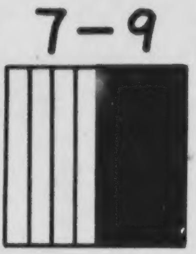
Several methods are possible. Invite two or three students to demonstrate their methods, and accept one of the following three partitions to draw on the board.



or



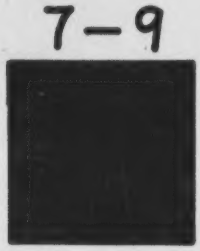
or



T: How should we color these four regions?

S: Color any three regions blue and color one region red.

Invite a student to do this coloring. Perhaps the following picture will result.



T: I want to divide the other half of the square into the same size pieces. Into how many pieces should I divide it?

S: Four.

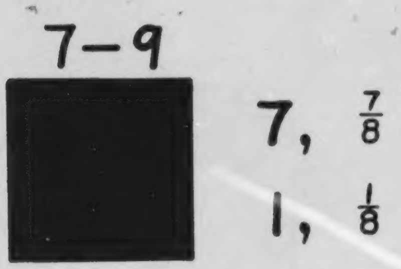
Divide that region in the same way the first half was divided.

T: What are Rita and Bruce's winning chances when the score is 7-9?

S: Bruce has 7 winning chances and Rita has 1.

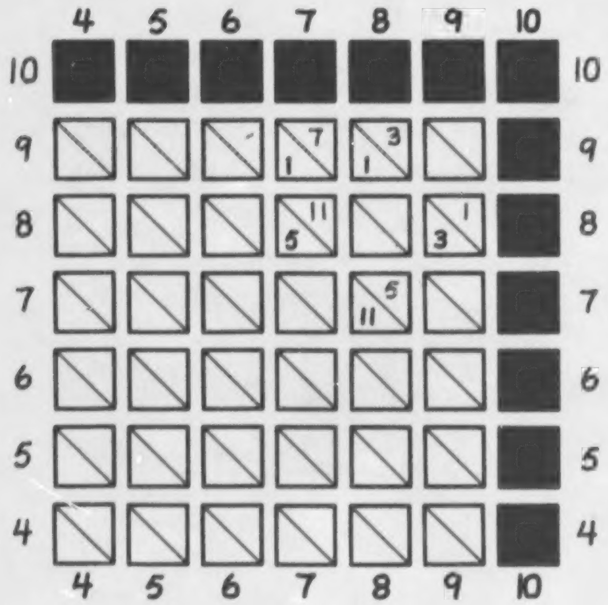
S: Bruce has 7 chances out of 8 to win and Rita has 1 chance out of 8.

Near the square, write both ways of recording these results.



P 4

Ask the students to record this result for 7-9 on their worksheets and invite a student to write it on Poster #2.

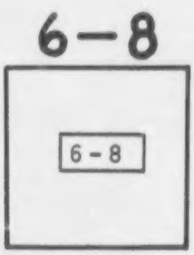


T: What other result can we record since we know the result for 7-9?

S: For 9-7, Rita has 7 winning chances and Bruce has 1.

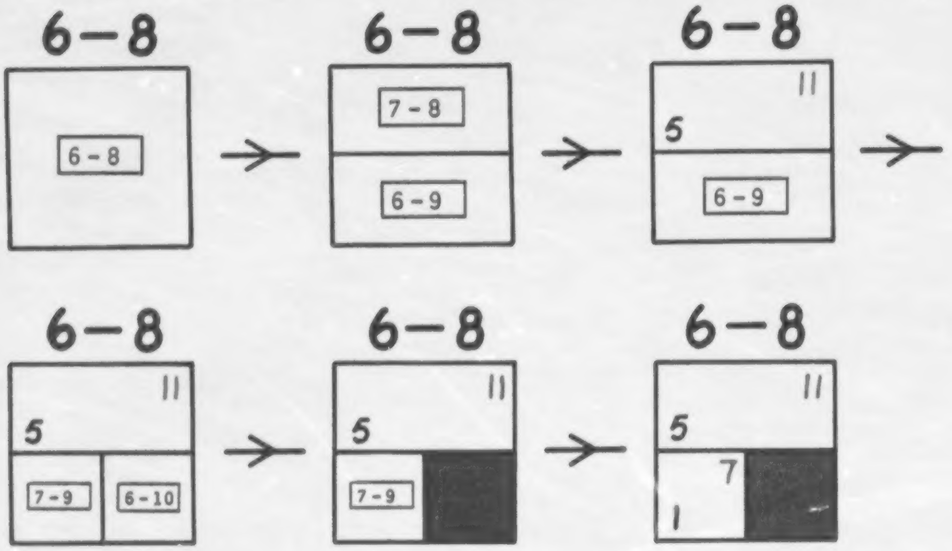
Ask the class to record this result for 9-7 on their worksheets and invite a student to write the result on Poster #2.

Erase the board and draw this square.



T: Copy this square and use it to calculate Rita's and Bruce's chances to win when the score is 6-8.

Let the students work independently for a while and help any student who needs it. Encourage students to use the results they already know. Then, following student suggestions, solve the problem collectively at the board. The sequence of squares drawn below show the steps your class might suggest.



T: It should not be necessary to actually divide each region into smaller pieces, but let's imagine that we do it. How would I divide and color each region?

S: You would divide the region with "5" and "11" into sixteen equal-sized pieces and color eleven blue and five red.

S: You would divide the region with "1" and "7" into eight equal-sized pieces and color seven blue and one red.

S: Divide the blue region into eight pieces because that region is the same size as the region with "1" and "7".

Write "8" near the blue region.

T: If I divided the regions in this way, would all the small pieces be the same size?

P 4

Let the class discuss this question until they agree that all the pieces would be the same size.

T: For the whole square, how many small pieces would be red?

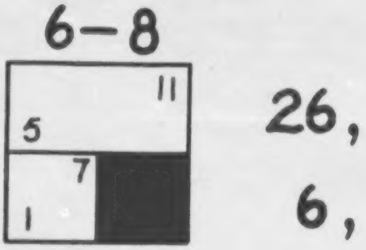
S: 6 (5 + 1 = 6).

T: How many would be blue?

S: 26 (11 + 7 + 8 = 26).

T: Therefore, when the score is 6-8, Bruce has 26 chances to win and Rita has 6 chances.

Write "26" and "6" near the square on the board.



T: How could we record these results as fractions to show Rita's and Bruce's probabilities of winning?

S: Write "$\frac{26}{32}$" for Bruce since he has 26 chances out of 32 to win. And write "$\frac{6}{32}$" for Rita since she has 6 chances out of 32 to win.

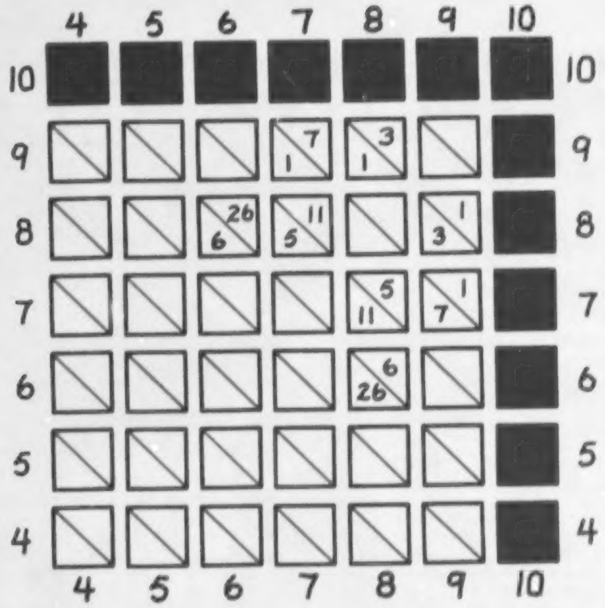
Write "$\frac{26}{32}$" and "$\frac{6}{32}$" near the square.

T: Do we also know the winning chances when the score is 8-6?

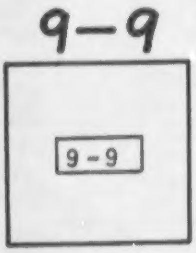
S: For 8-6, Rita has 26 chances and Bruce has 6 chances.

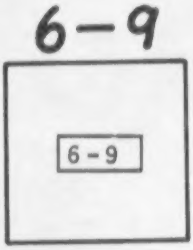
S: For 8-6, Rita's probability of winning is $\frac{26}{32}$ and Bruce's probability of winning is $\frac{6}{32}$.

Instruct students to record these results for 6-8 and 8-6 on their worksheets, and invite a student to record these results on the demonstration poster.



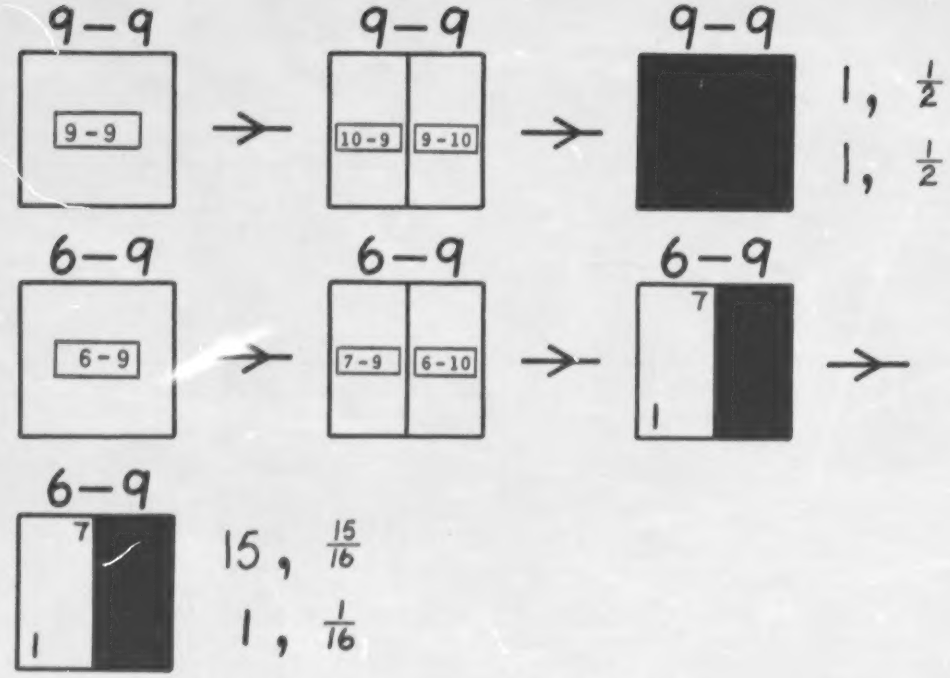
On the board, draw the two squares shown below and ask the class to find Rita and Bruce's winning chances for each score.



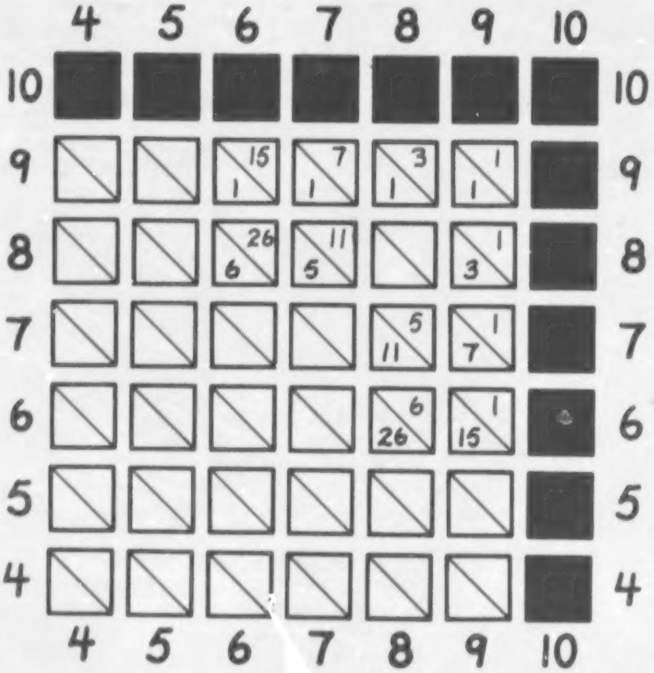


Assist any student needing help. One method of solution for each problem is indicated in the next illustration. Do these two problems collectively only if many students are still having difficulty.

P 4


9–9
9-9
9–9
10-9
9-10
9–9
1, 1/2
1, 1/2
6–9
6-9
6–9
7-9
6-10
6–9
7
1
6–9
7
1
15, 15/16
1, 1/16

Invite students to record the results for 9-9, 6-9, and 9-6 on Poster #2. Instruct the students to also record these results on their worksheets.



	4	5	6	7	8	9	10	
10	■	■	■	■	■	■	■	10
9			15 / 1	7 / 1	3 / 1	1 / 1	■	9
8			26 / 6	11 / 5		1 / 3	■	8
7					5 / 11	1 / 7	■	7
6					6 / 26	1 / 15	■	6
5							■	5
4							■	4
	4	5	6	7	8	9	10	

T: We could continue in the same manner to complete this chart, but that would be very time consuming. Instead let's try to find some patterns among the numbers on the chart and then use these patterns to complete the chart.

Call the class' attention to the top row of results on the chart.

T: We have Rita's and Bruce's winning chances for the scores 9-9, 8-9, 7-9, and 6-9. Can you find a pattern to help calculate their chances when the score is 5-9? How many chances to win do you think Rita has when the score is 5-9?

S: 1, because each red number in the top row is 1.

Write "1" in the square for 5-9 on Poster #2.

T: The numbers written in blue in the top row are 1, 3, 7, and 15. Do you notice a pattern? What number should we write in blue for the score 5-9? Write the number on your paper.

Let your students look for the pattern for a few minutes. As you walk around the class, instruct students who are correct to fill in the 5-9 square on their worksheet and help any students who need it.

T: What are Bruce's chances when the score is 5-9?

S: 31.

T: What pattern did you use?

Elicit explanations from several students if they noticed different patterns.

S: To calculate each number in the sequence, 1, 3, 7, 15, multiply the previous number by 2 and add 1. For example, (2 x 1) + 1 = 3, (2 x 3) + 1 = 7, and (2 x 7) + 1 = 15. Therefore the next number is 31 since (2 x 15) + 1 = 31.

P 4

S: I noticed that 1 + 2 = 3, 3 + 4 = 7, and 7 + 8 = 15. The number you add each time, 2, 4, 8, is doubled. The next number should be 31, since 15 + 16 = 31 and 16 is the double of 8.

S: The sum of the red and blue numbers in each square doubles each time, 1 + 1 = 2, 1 + 3 = 4, 1 + 7 = 8, and 1 + 15 = 16. Since 2 x 16 = 32 and 1 + 31 = 32, the next number should be 31.

Write " 31 " in the square for 5-9 on Poster #2. Invite a student to also record the result for 9-5.

T: Use one of these patterns to calculate the chances when the score is 4-9. What number should we write in red?

S: 1.

Write " 1 " in the square for 4-9 on Poster #2.

T: What number should we write in blue for the score 4-9?

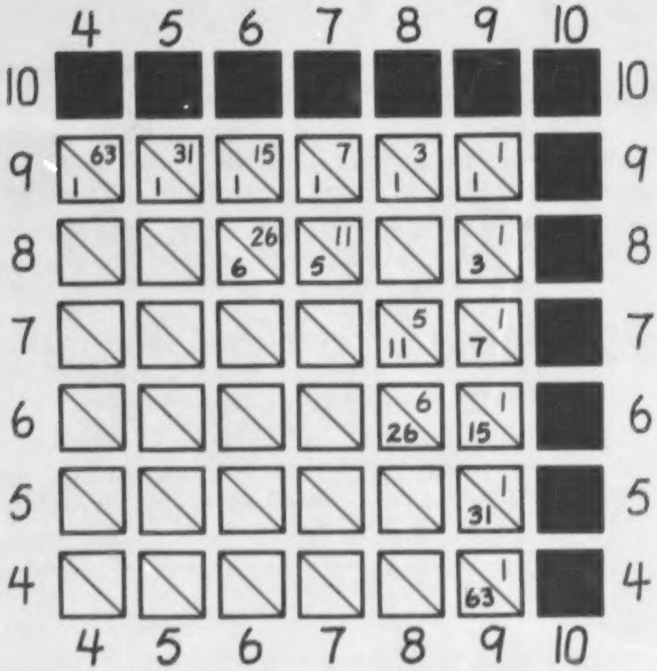
Encourage several explanations if students are using different patterns.

S: 63, since (2 x 31) + 1 = 63.

S: 63, since 31 + 32 = 63 and 32 is the double of 16.

S: 63, since 1 + 63 = 64 and 64 = 2 x 32.

Write " 63" in the square for 4-9 on Poster #2 and invite a student to also record the result for 9-4. Instruct the students to check that they have copied all the results from the poster onto their worksheets.



	4	5	6	7	8	9	10	
10	■	■	■	■	■	■	■	10
9	63 1	31 1	15 1	7 1	3 1	1 1	■	9
8			26 6	11 5		1 3	■	8
7					5 11	1 7	■	7
6					6 26	1 15	■	6
5						1 31	■	5
4						1 63	■	4
	4	5	6	7	8	9	10	

T: In the next lesson, we will try to find another pattern which will let us quickly complete the rest of the chart. By studying the poster and thinking about how we have calculated the chances, you might be able to guess this pattern already.

Collect Worksheets P07 and P08. The worksheets and both posters will be used in Lesson P3.3.

P 5

2.3 Probability Games #3

CAPSULE LESSON SUMMARY

Review the probability game from Lessons P2.1 and P2.2 and recall how to read the posters. Investigate some patterns in the chart on Poster #2 and use these patterns to complete the chart. Use the completed chart to discuss and solve some problems about related probability games.

MATERIALS

Teacher: Probability Posters #1 and #2 for IG-IV; Scorecards for Probability Games; colored chalk

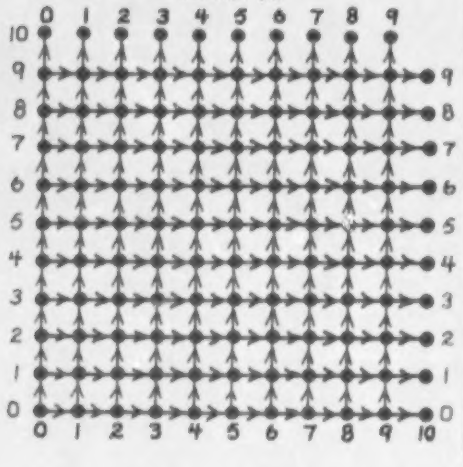
Student: Paper; pencil; Worksheets P08, P09

NOTE: Scorecards should be prepared as described for Lesson P2.1 so that they stick to the board. Worksheet P09 is a partial answer key for Worksheet P08 and is not needed until Exercise 3. Students should not be looking at Worksheet P09 until that time.

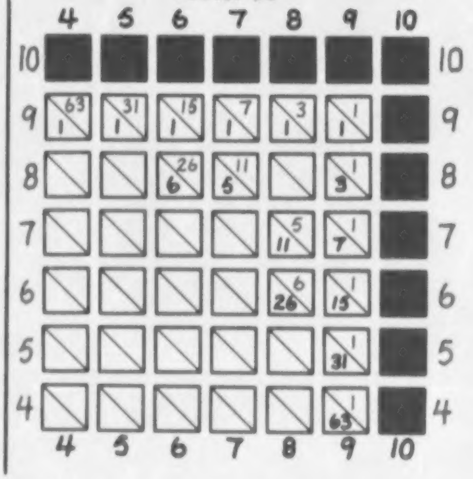
DESCRIPTION OF LESSON

Display Probability Posters #1 and #2 for IG-IV at opposite ends of the board. The results from Lessons P2.1 and P2.2 should still be recorded on Poster #2. Display the scorecards near the center of the board.

Poster #1


0 1 2 3 4 5 6 7 8 9
10 9 8 7 6 5 4 3 2 1 0
9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10

Poster #2


4 5 6 7 8 9 10
10 9 8 7 6 5 4
63 1
31 1
15 1
7 1
3 1
1 1
26 6
11 5
1 3
5 11
1 7
6 26
1 15
1 31
1 63
4 5 6 7 8 9 10

4-8 | 5-8 | 6-8 | 7-8 | 8-8 | 9-8 | 10-8

4-9 | 5-9 | 6-9 | 7-9 | 8-9 | 9-9 | 10-9

4-10 | 5-10 | 6-10 | 7-10 | 8-10 | 9-10

Exercise 1

Distribute the students' copies of Worksheet P08 which were collected at the end of Lesson P2.2.

T: Who remembers the game that Rita and Bruce were playing?

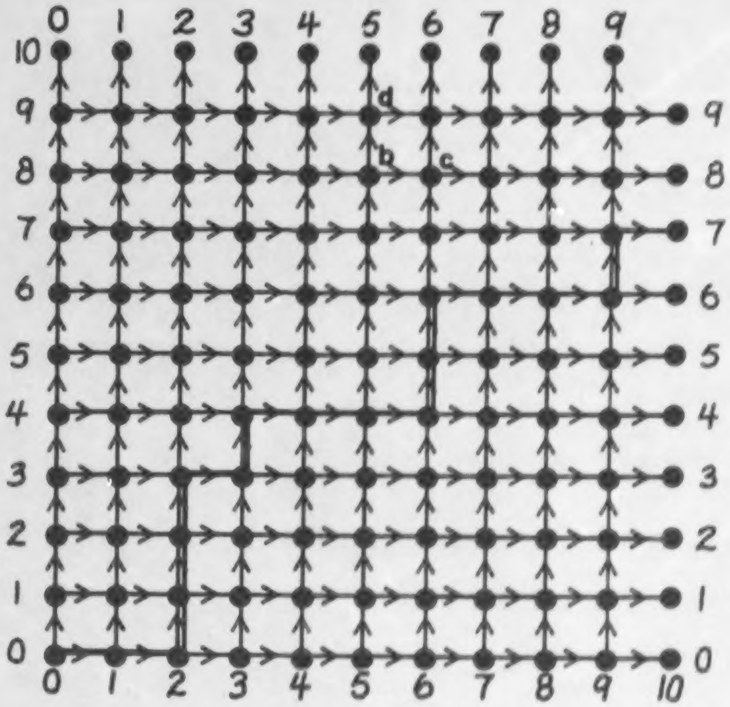
Make certain that at least the following aspects of the game are mentioned:

- one red marble and one blue marble are used;
- Rita scores a point if the red marble is chosen and Bruce scores a point if the blue marble is chosen;
- the first player to score ten points wins.

Call the class' attention to Poster #1. Point to the dot for the score 0 - 0.

T: Each game that Rita and Bruce play starts at the score 0 - 0 (read this score as "0 to 0"). Who can trace a path of red and blue arrows to show any game that is a win for Rita?

The path traced by a student must start at 0 - 0, follow blue arrows upward or red arrows to the right, and end at a red dot in the rightmost column. For example, a student might trace the path shown in black in the next illustration.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Who can trace a path to represent a shortest possible game?

The path from 0 - 0 to 10 - 0 with ten red arrows and the path from 0 - 0 to 0 - 10 with ten blue arrows represent the two shortest games. Encourage the class to find both of these paths.

T: Who can trace a path to represent a longest possible game? How many arrows are there in a path for a longest game?

Any path from 0 - 0 to 10 - 9 or from 0 - 0 to 9 - 10 has exactly 19 arrows and represents a longest possible game.

T: Which dot is for the score 5 - 8?

A student should point to b.

T: If the score in a game is 5 - 8, what could the next score be?

S: 6 - 8 or 5 - 9.

Use the grid to check this answer.

T: The score changes from 5 - 8 to 6 - 8 if the red marble is chosen (trace a red arrow from b to c). The score changes from 5 - 8 to 5 - 9 if a blue marble is chosen (trace a blue arrow from b to d).

Call the class' attention to the chart on Poster #2.

T: We have already filled in part of this chart. What are Rita's and Bruce's chances to win when the score is 8 - 6?

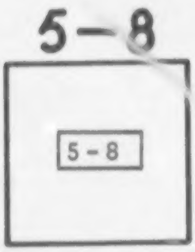
S: Rita has 26 chances to win and Bruce has only 6 chances.

T: What are their chances when the score is 5 - 9?

S: Bruce has 31 chances and Rita has 1 chance.

T: In a previous lesson, we found a pattern we could use to fill in the squares in the top row and rightmost column. Today we will try to find a pattern to use to fill in the rest of the chart.

Draw a large square on the center board and label it "5 - 8".



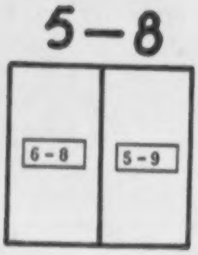
P 5

T: Let's calculate Rita's and Bruce's winning chances when the score is 5 - 8. If the score is 5 - 8, what could the next score be?

S: 6 - 8 or 5 - 9.

T: How do I show this on the square?

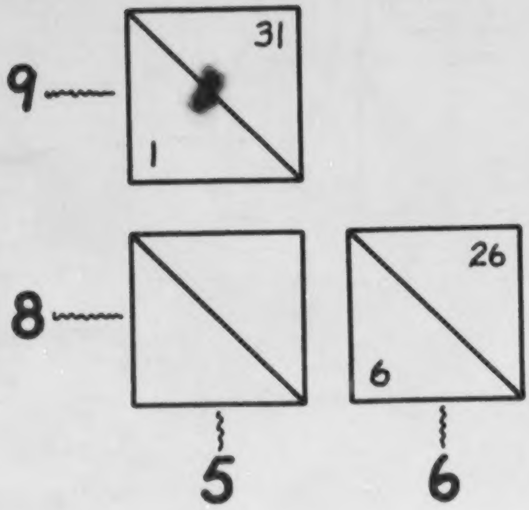
S: Divide the square in half. Put $\boxed{6-8}$ in one half and $\boxed{5-9}$ in the other half.



T: In order not to do too much work, we should check whether we know the results for the scores 6 - 8 and 5 - 9. Have we filled in the squares for these two scores on our chart?

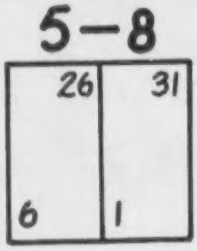
S: Yes, if the score is 6 - 8, Bruce has 26 chances to win and Rita has 6 chances. If the score is 5 - 9, Bruce has 31 chances and Rita has 1 chance.

The next illustration is a picture of part of the chart on Poster #2. Do not draw this on the board, but refer to this part of the chart as you point out the relationships among the squares for 5 - 8, 6 - 8, and 5 - 9.



T (pointing to the appropriate squares in turn): To figure out the chances for 5 - 8, we can use the results recorded in this square for 6 - 8 and the results recorded in this square for 5 - 9.

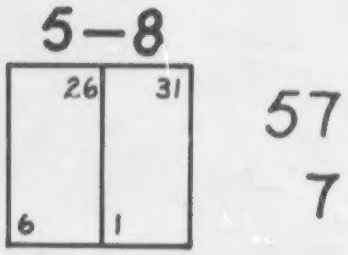
Write the results for 6 - 8 and for 5 - 9 in their respective regions of the large square on the board.



T: What are the winning chances for Bruce and for Rita when the score is 5 - 8?

S: 57 for Bruce and 7 for Rita.

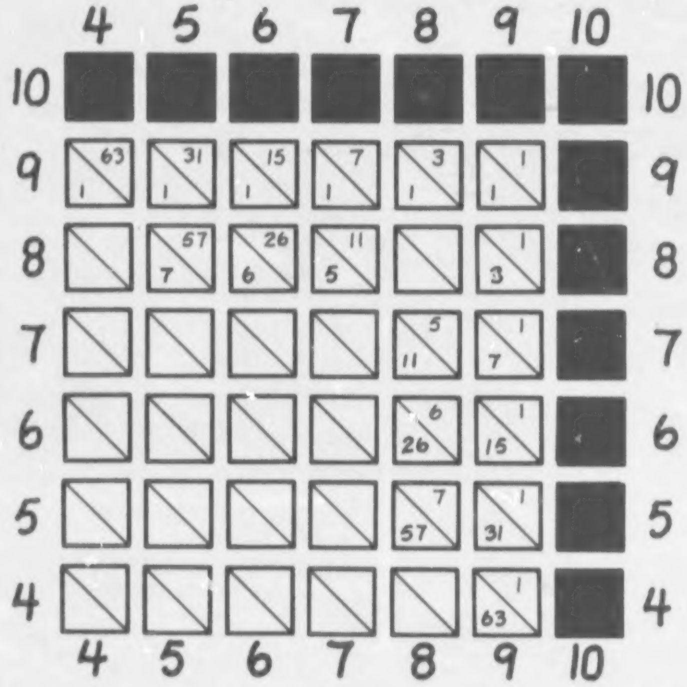
P 5



T: Which other result do we know immediately?

S: When the score is 8 - 5, Rita has 57 chances to win and Bruce has 7.

Instruct the students to record these results for 5 - 8 and 8 - 5 on their worksheets. Invite a student to fill in the same squares on Poster #2.



	4	5	6	7	8	9	10	
10	■	■	■	■	■	■	■	10
9	63 1	31 1	15 1	7 1	3 1	1 1	■	9
8		57 7	26 6	11 5		1 3	■	8
7					5 11	1 7	■	7
6					6 26	1 15	■	6
5					7 57	1 31	■	5
4						1 63	■	4
	4	5	6	7	8	9	10	

T: Let's try to find some patterns among the results on the poster and predict the winning chances for Rita and Bruce when the score is 4 - 8 (point to the square for 4 - 8). How many winning chances do you think Rita has when the score is 4 - 8?

S: 8. Rita's chances for 7 - 8, 6 - 8, and 5 - 8 are 5, 6, and 7 respectively. Therefore, she should have 8 winning chances when the score is 4 - 8.

On the board, write "8" and any other predictions for Rita's chances when the score is 4 - 8.

T: Can you predict how many winning chances Bruce has when the score is 4 - 8?

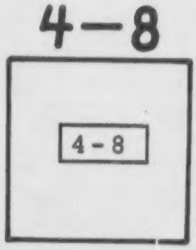
Let the students consider this problem for a while and ask them to write their predictions on a piece of paper. Then invite students to give their predictions and explain the patterns they used. Without revealing who is correct, record the predictions on the board. Two of the patterns which give the correct answer of 120 are explained below. Do not be concerned if no one predicts 120, since this answer will be found collectively and compared to the predictions.

S: I noticed that Bruce's 26 winning chances for 6 - 8 can be calculated by adding 11 and 15, his chances for 7 - 8 and 6 - 9 respectively. His 57 chances for 5 - 8 can be calculated by adding 26 and 31, his chances for 6 - 8 and 5 - 9 respectively. Therefore his chances for 4 - 8 can be found by adding 57 and 63, his chances for 5 - 8 and 4 - 9 respectively. 57 + 63 = 120, therefore, Bruce's chances for 4 - 8 should be 120.

S: Bruce's winning chances for 7 - 8, 6 - 8, and 5 - 8 are 11, 26, and 57 respectively. I noticed that (2 x 11) + 4 = 26 and (2 x 26) + 5 = 57. Therefore, his chances for 4 - 8 should be (2 x 57) + 6 = 120.

P 5

Draw this square on the board.

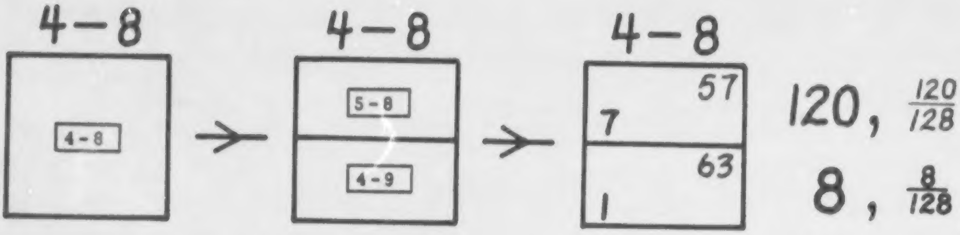


T: Let's check our predictions by using this square to calculate Rita and Bruce's winning chances when the score is 4 - 8. Can we use some results that we have already calculated?

S: We could use the results we know for 5 - 8 and 4 - 9. If the score is 4 - 8, the next score will be 5 - 8 or 4 - 9.

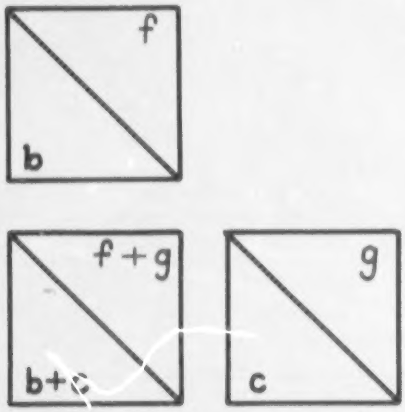
T: Copy this square for 4 - 8, and use it to calculate Rita's and Bruce's winning chances when the score is 4 - 8.

Give the students a few minutes to solve the problem for 4 - 8. Individually, help students who are having difficulty. Then solve the problem collectively at the board. The following illustration indicates steps that could be followed.



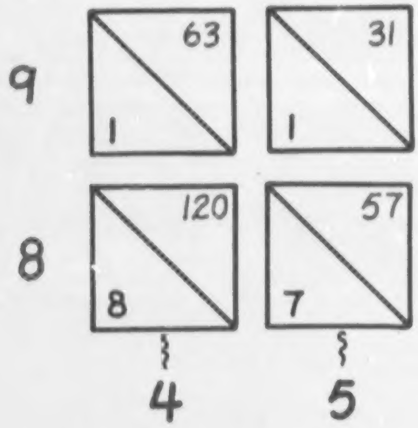
Observe that anyone who predicted 120 for Bruce and 8 for Rita was correct. Invite a student to record the results for 4 - 8 and 8 - 4 on Poster #2. Instruct students to write the results on their worksheets.

NOTE: The goal in the next part of the lesson is to lead the students to observe a pattern which will allow them to quickly complete the rest of Poster #2. For your reference only, the following illustration uses symbols to concisely explain this pattern to you.



To find the result for the lower left square when the results for the squares directly above and to the right are already known, simply add the numbers as indicated in the above illustration.

Refer to the following part of Poster #2.



P 5

Point to "120" in the square for the score 4 - 8.

T: How did we calculate to get 120 here?

S: We added 57 and 63.

T: Where are "57" and "63" on the poster?

Let a student point out that "57" is written in the square directly to the right of the square with "120", and "63" is in the square directly above the square with "120".

T (pointing to "8" in the square for 4 - 8): Does the same pattern work for this "8"?

S: Yes, 8 = 7 + 1. "7" and "1" are red in the same squares as "57" and "63" are blue, respectively.

T: Does this pattern work with the other results we have already calculated?

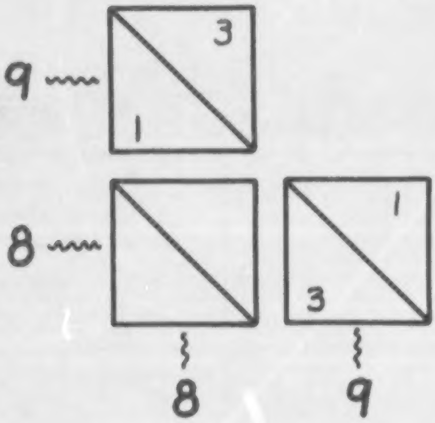
The students should observe that this pattern works for the scores: 5 - 8, 6 - 8, 8 - 4, 8 - 5 and 8 - 6 that have already been recorded.

T: Let's use this pattern to calculate the results for some more scores. For example, can you figure out Rita's and Bruce's winning chances when the score is 8 - 8?

S: Rita has 4 chances to win and Bruce has 4 chances to win.

T: Why?

Direct a student to point to the appropriate part of the poster.



S: Add 3 and 1 to get Bruce's winning chances for 8 - 8, and add 1 and 3 to get Rita's winning chances for 8 - 8.

Write "4" and "4" in the square for 8 - 8.

A student might observe that when the score is tied at 8 - 8, Rita and Bruce have the same chances to win and we could write "1" for Rita and "1" for Bruce. If a student suggests this, agree that it is a correct answer, but note that this formulation of the answer does not follow the pattern just discovered. Suggest that the class try to express the answer so that it does follow the pattern; i.e., 4 chances for Rita and 4 chances for Bruce.

T: Why do Rita and Bruce have the same number of winning chances, namely 4, when the score is 8 - 8?

S: The game is tied at 8 - 8. Rita and Bruce must each have the same number of chances to win.

Invite the class to use the pattern to quickly calculate the results for several other scores such as 7 - 7, 6 - 7, 5 - 7, 4 - 7, 7 - 6, and 6 - 6. Use the chart as shown on the next page as an answer key. When you feel that most of the class understands the pattern, instruct the students to use the pattern to complete their worksheets. Assist any students needing help. As the students work on this, gradually invite students with correct answers to complete the chart on the poster.

	4	5	6	7	8	9	10	
10	■	■	■	■	■	■	■	10
9	63 1	31 1	15 1	7 1	3 1	1 1	■	9
8	120 8	57 7	26 6	11 5	4 4	1 3	■	8
7	219 37	99 29	42 22	16 16	5 11	1 7	■	7
6	382 130	163 93	64 64	22 42	6 26	1 15	■	6
5	638 386	256 256	93 163	29 99	7 57	1 31	■	5
4	1,024 1,024	386 638	130 382	37 219	8 120	1 63	■	4
	4	5	6	7	8	9	10	

Continue the collective lesson once Poster #2 is filled in correctly.

Exercise 2

T: Now that you have discovered the pattern, let's use our completed chart to solve some related problems concerning this game.

Write this information on the board.

9 – 7: 7 1

8 – 6: 26 6

T: Rita is favored for both of the scores 9 - 7 and 8 - 6. If you were Bruce, would you prefer the score to be 9 - 7 or 8 - 6?

S: 8 - 6, because then Bruce would have more time to catch up. For both scores, Bruce is two points behind Rita.

S: 8 - 6, because if the score is 9 - 7, Rita is seven times more likely to win then Bruce since 7 = 7 × 1. But if the score is 8 - 6, Rita is only about four times more likely to win since 26 is about 4 × 6.

Erase the board and write this information.

(100) 97–96

T: Rita and Bruce are now playing the same game, but the first to score 100 points wins. If the score is 97 - 96, can you use the same chart to find Rita's and Bruce's chances to win?

S: When the score is 97 - 96 in a game to 100, Rita needs three points to win and Bruce needs four points to win. This situation is the same as when the score is 7 - 6 in the game to 10; i.e., Rita needs three points and Bruce needs four points. Therefore, when the score is 97 - 96, Rita has 42 chances to win and Bruce has 22.

If no one gives an explanation similar to this, ask the following questions.

T: If the score is 97 - 96 and the goal is 100, how many points does Rita need to win?

S: Three.

T: How many points does Bruce need to win?

S: Four.

P 5

T: In their old game to ten points, is there ever a situation when Rita needs three points to win and Bruce needs four points to win?

S: Yes, when the score is 7 - 6.

T: Therefore, a score of 97 - 96 in a game to 100 is similar to a score of 7 - 6 in a game to 10. What are Rita's and Bruce's chances when the score is 97 - 96?

S: The same as when the score is 7 - 6 in their old game. Rita has 42 winning chances and Bruce has 22 winning chances.

Write this information on the board.

㉕ 22 – 24

T: On another day, Rita and Bruce set a goal of 25. The score in their game is 22 - 24. What are Rita's and Bruce's chances to win?

S: 7 for Bruce and 1 for Rita.

T: Why?

S: When the score is 22 - 24, Rita needs three points to win and Bruce needs one point. That is the same as when the score is 7 - 9 in the game to 10. And for 7 - 9, Bruce has 7 chances and Rita has 1 chance.

Write this information on the board.

Rita	⑮	10
Bruce	⑫	10

T: One day, Rita was winning more often than Bruce. They decided that in their next game Rita would have to score 15 to win and Bruce would only need to score 12. If the score is 10 - 10, what are their chances of winning?

S: 57 for Bruce and 7 for Rita.

T: Why?

S: When the score is 10 - 10 in this game, Rita needs five points to win and Bruce needs two points to win. That is the same as when the score is 5 - 8 in the game to 10. And when the score is 5 - 8, Bruce has 57 chances to win and Rita has 7.

Exercise 3

This exercise on number patterns in the chart is optional. Explore these patterns as time allows until the end of the period.

Distribute Worksheet P09.

T: Let's try to find some more patterns on this chart. Worksheet P09 has only the numbers in blue so that it is easier to see the patterns.

Write this series of numbers on the board.

☐
7
11
16
22
29
37
☐

P 5

Point to the vertical column labeled " 7 " on Poster #2.

T: These numbers are from the column labeled "7". Try to find a pattern to this sequence of numbers. According to your pattern, what numbers should be written in the two boxes? Write your answers on a piece of paper.

Walk around and check the answers of a few students.

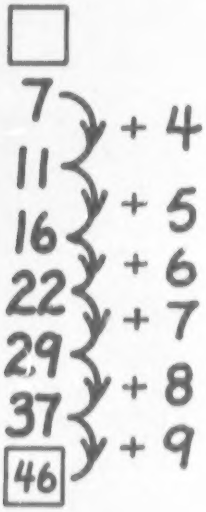
T: What number should come after 37 in this sequence?

S: 46.

T: What pattern did you use?

S: I saw that 7 + 4 = 11, 11 + 5 = 16, 16 + 6 = 22, 22 + 7 = 29, and 29 + 8 = 37. The number you add on is one larger each time. The next number should be 46 because 37 + 9 = 46.

Add this information to your picture.



T: What number should come before 7?

S: 4.

T: Why?

S: You must add three to some number to get seven. 4 + 3 = 7.

Erase the board and write this sequence of numbers.

$$\square,\ \square,\ 120,\ 57,\ 26,\ 11,\ 4,\ 1$$

T: These numbers are from the horizontal row labeled "8". Can you find a pattern?

S: Starting from 1 at the right, you double 1 and add 2 to get 4. Then double 4 and add 3 to get 11. Then double 11 and add 4 to get 26. This goes on; each time you double the number and add one more than you did the last time.

S: (1 × 2) + 2 = 4 and (4 × 2) + 3 = 11 and (11 × 2) + 4 = 26 and (26 × 2) + 5 = 57 and (57 × 2) + 6 = 120.

Following your students' suggestions, add this information to your picture.

$$\square,\ \square,\ \underset{(57\times2)+6}{120},\ \underset{(26\times2)+5}{57},\ \underset{(11\times2)+4}{26},\ \underset{(4\times2)+3}{11},\ \underset{(1\times2)+2}{4},\ 1$$

T: On your paper, write what the next two numbers should be.

After a short time, ask for these numbers.

S: 247 comes next because (120 × 2) + 7 = 247.

S: 502 comes after 247 because (247 × 2) + 8 = 502.

Name ____________________ P 08 *

	4	5	6	7	8	9	10	
10	■	■	■	■	■	■	■	10
9	63 1	31 1	15 1	7 1	3 1	1 1	■	9
8	120 8	57 7	26 6	11 5	4 4	1 3	■	8
7	219 37	99 29	42 22	16 16	5 11	1 7	■	7
6	382 130	163 93	64 64	22 42	6 26	1 15	■	6
5	638 386	256 256	93 163	29 99	7 57	1 31	■	5
4	1,024 1,024	386 638	130 382	37 219	8 120	1 63	■	4
	4	5	6	7	8	9	10	

383